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CONVEX FIGURES AND POLYHEDRA

T O P I C S   I N   M A T H E M A T I C S

Convex Figures and Polyhedra

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Translated and adapted from the first Russian edition (1956) by

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SURVEY OF

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A project conducted by

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PREFACE TO THE AMERICAN EDITION

THE THEORY of convex figures and polyhedra provides an excellent example of a body of mathematical knowledge that offers theorems with elementary formulations and vivid geometric meaning. Despite this simplicity of formulation, the proofs are often not elementary. Thus the area presents a particular challenge to mathematicians, who have investigated convex figures and polyhedra for millenia, and yet have by far not exhausted the subject. Many of the theorems in this volume were in fact proved only a few years ago.

The material in this book will be suitable for study in mathematics clubs or by readers with a background of secondary school mathematics only. The topics considered are stimulating and challenging, and moreover, convexity ideas are valuable in the study of modern higher mathematics. Mathematical analysis, higher geometry, and topology each use convexity notions in an essential way.

Chapter 1 presents basic information on convex figures and bodies, and on their supporting lines and planes. Most of the exposition is quite elementary. Minkowski's theorem on ovals of constant width is treated here, and a less elementary question about maxima and minima is discussed.

Chapter 2, also elementary for the most part, discusses some properties of central-symmetric polyhedra and Minkowski's theorem on the largest central-symmetric solid in the lattice of integers.

Chapter 3 is devoted to the fundamental theorems on convex polyhedra (Chapter 5 is similar in content). This material does not require a knowledge of higher mathematics, although for Sections 14–17 the reader should have some experience in reading mathematical literature. The formulation of A. D. Aleksandrov's theorem on the development of a convex polyhedron is given in Section 18.

Chapter 4, unlike the previous chapters, requires some familiarity with elementary analytic geometry and integral calculus. It gives the elementary theory of linear systems of convex figures for the case of the plane.

Chapter 5, written by A. D. Aleksandrov, contains a proof of his theorem on convex polyhedra. Minkowski's theorem stating that a convex polyhedron is completely determined by the areas and direc-

tions of its faces is a special case of this theorem. The proof given uses only elementary methods, and so Minkowski's theorem has now been included in the elementary mathematical literature.

Chapter 6 gives more precise definitions and generalizations of concepts discussed earlier in the book, for example, those of convex figures. It also generalizes some of the material in earlier chapters, presents new material in a manner similar in geometric style to the exposition in the earlier chapters, and touches upon important theorems that give some idea of the connection between topology and the theory of convex figures.

For a further discussion of convex figures and for problems and exercises in this theory, the reader may wish to consult I. M. Yaglom and V. G. Boltyanskii, *Convex Figures,* Holt, Rinehart and Winston, 1961.

The Survey of Recent East European Mathematical Literature wishes to express its appreciation to Mr. Donald L. Barnett for the translation and for his valuable revisions and improvements of the text.

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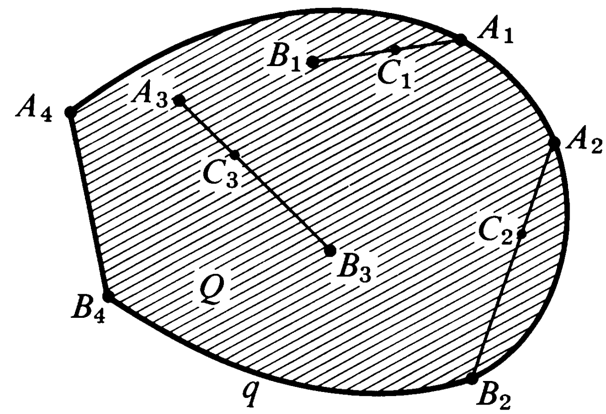
1. Convex Figures

1. PLANE CONVEX FIGURES

The reader has already met some convex figures (namely, convex polygons) in a course in elementary geometry. We now give some formal definitions.

DEFINITION. *A figure is called a* convex figure *if it contains the whole of every line segment joining two of its points* (Fig. 1).[1]

Fig. 1



Examples of plane convex figures are the circle, the semicircle, and the ellipse.[2] All triangles are convex. Some quadrilaterals are convex (for example, the parallelogram), and some are not convex (Fig. 2). A circular sector is a convex figure if its central angle is less than π radians (180°), but not convex if its central angle is greater than π radians.

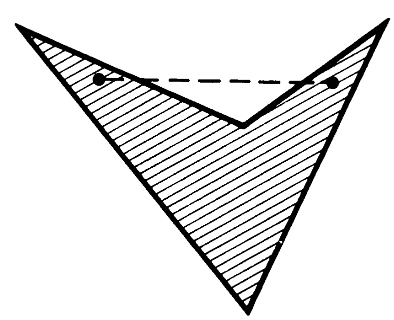


Fig. 2

DEFINITION. *A plane convex figure is called* bounded *if it can be placed inside a circle with finite radius; otherwise it is called* unbounded.

All the above-mentioned convex figures are bounded. On the other hand, every plane is an unbounded convex figure. A straight

[1] A rigorous definition of *convex figure* will be given in section 37. (The reader should notice that in the Russian usage the term *figure* includes the region enclosed by the curve, as well as the curve itself.)

[2] Examples of convex figures in three-dimensional space will be given in section 7.

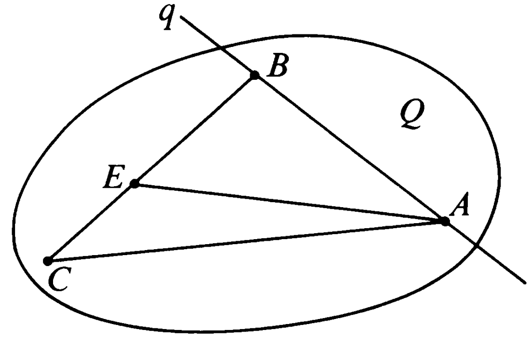
line divides a plane into two *half-planes.* A half-plane is an unbounded convex figure, as is a strip between two parallel lines. Two *half-lines* originating in one point and not lying in one straight line divide the plane into two parts (two angles). One of these is less than π radians and is an unbounded convex figure; the other is greater than π radians and is not a convex figure.

A straight line is a convex figure, since for any two of its points, A and B, it contains the entire line segment, AB. Similarly, any half-line or any line segment is a convex figure. Conversely, any convex part of a straight line (other than the whole line) is a line segment or a half-line.

DEFINITION. *A* convex polygon *is a plane convex figure whose boundary consists of several line segments* (sides *of the polygon*). *The points where the ends of two neighboring sides meet are called* vertices *of the polygon.*

Now suppose that Q is a convex figure not lying on one straight line (Fig. 3). Let A and B be two of its points, and q be the line contain-

Fig. 3



ing them. Since Q does not lie entirely on q, the figure Q contains a point, C, not lying on q; that is, Q contains three points, A, B, and C, not lying on one line. Now we can show that Q contains the whole triangle ABC as follows:

THEOREM 1. *If a convex figure Q contains three points, A, B, and C, not lying on one line, then Q contains the whole triangle ABC.*[1]

Proof. Since Q is a convex figure containing the points A, B, and C (Fig. 3), it must contain all three sides, AB, AC, and BC, of triangle ABC. Triangle ABC is covered[2] by the line segments AE connecting the vertex A with all possible points E of the opposite side BC. Since the ends of the segments AE belong to figure Q, then every segment AE belongs entirely to figure Q.

[1] This theorem is true for space as well as for the plane.

[2] See footnote 1 on page 1.

Corollary 1. *The smallest convex figure containing three given points A, B, and C not lying on one line is the triangle ABC.*

Corollary 2. *A curve different from a straight line, a half-line, or a line segment cannot be a convex figure.*

Thus we are satisfied that a convex figure either is linear (that is, a straight line, a half-line, or a line segment) or is not linear (that is, contains a triangle).

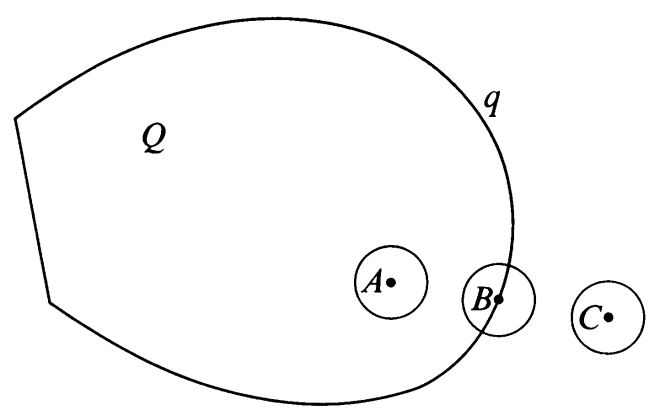
The straight line, the half-line, and the line segment are called *one-dimensional* convex figures. Plane convex figures not lying on one straight line are called *two-dimensional.* We shall also consider a point as a convex figure and call it a *zero-dimensional* convex figure.

If Q is a two-dimensional convex figure, then with respect to Q, all points of the plane are divided as follows:

(1) Points *exterior to Q*. Around any exterior point it is possible to draw a circle lying entirely outside of Q (point C in Fig. 4).

(2) Points *belonging to Q*. These are divided as follows:
   (*a*) *Interior* points of Q. Around any interior point it is possible to draw a circle lying entirely in Q (point A in Fig. 4).
   (*b*) *Boundary* points of Q. Every circle drawn around a boundary point contains both interior and exterior points of Q (point B in Fig. 4).

Fig. 4



Definition. *The set of interior points of Q is called the* interior *of Q, and the set of boundary points of Q is called the* boundary of Q. *The boundary of a convex two-dimensional figure Q is a curve q called a* convex curve. (However, a convex curve is not a *convex figure,* according to the definition of the latter.)

THEOREM 2. *The line segment connecting an interior point A of a convex figure Q with any other point B of the same figure is entirely (with the possible exception of point B) made up of interior points of the figure.*

Proof. Since A is an interior point of figure Q (Fig. 5), it is possible to draw a circle with center at A belonging entirely to figure Q.

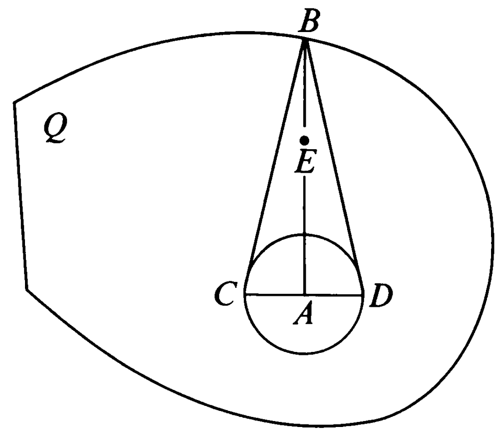


Fig. 5

Let CD be the diameter of this circle perpendicular to AB. Since points B, C, and D belong to Q, the entire triangle BCD belongs to Q. On line segment AB take an arbitrary point E, different from the ends A and B of this segment. Then E lies inside triangle BCD and, consequently, inside figure Q containing triangle BCD.

COROLLARY 1. *If points A and B lie inside Q, then the entire line segment AB lies inside Q.*

COROLLARY 2. *If A and B are points of the boundary q of a convex figure Q, then either the entire line segment AB belongs to the boundary q or the entire line segment AB, with the exception of its end points, lies inside Q.*

Proof. There are two possible cases:

Case 1. AB belongs entirely to the boundary q (A_4B_4 in Fig. 1).

Case 2. At least one point, C, of this line segment lies inside Q (A_2B_2 and point C_2 in Fig. 1). In this case, by Theorem 2, both line segments A_2C_2 and C_2B_2 (with the exception of points A_2 and B_2) lie inside Q, and so the segment A_2B_2 lies entirely, with the exception of its end points, inside Q.

2. INTERSECTIONS AND PARTITIONS OF PLANE CONVEX FIGURES

DEFINITION. *The* intersection *of two figures is the set of all points belonging simultaneously to both figures, or, more briefly, the common part of these figures. If the figures do not have a common point, then we say that their intersection is* empty *or that their intersection is the* empty set.

For example, the intersection of two different straight lines is either a point or the empty set; the intersection of a half-plane and a circle in the plane is the entire circle, a segment of it, a point, or the empty set.

THEOREM 3. *If the intersection of two convex figures is not empty, it is a convex figure.*

Proof. If the intersection of two convex figures Q and Q_1 is nonempty, the following cases are possible:

Case 1. Q and Q_1 have only one common point. In this case, the intersection is a zero-dimensional convex figure.

Case 2. The intersection of Q and Q_1 contains more than one point. Denote the intersection by Q_2. Let A and B be two arbitrary points of Q_2. Each of the two figures Q and Q_1 is convex; therefore, together with points A and B, each of the figures Q and Q_1 contains the entire line segment AB. Thus, segment AB belongs simultaneously to both Q and Q_1 and, consequently, to their intersection Q_2. Hence, Q_2 is a convex figure, and the theorem is proved.

If Q and Q_1 are two-dimensional convex figures, then the intersection, Q_2, may be zero-dimensional (q_0 in Fig. 6) or

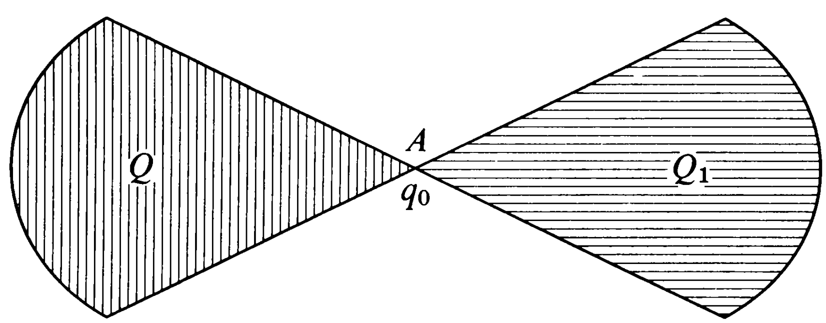


Fig. 6

Fig. 7

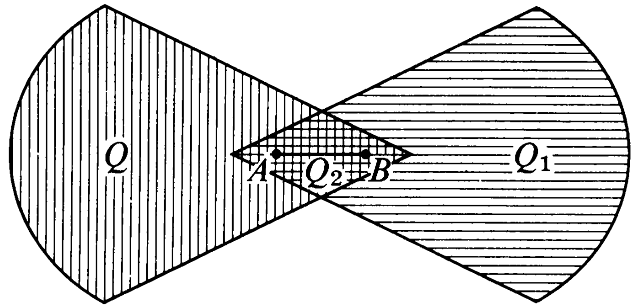
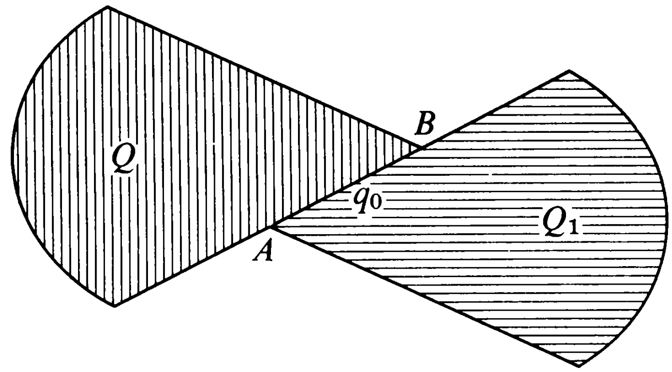


Fig. 8



two-dimensional (Q_2 in Fig. 7) or one-dimensional (q_0 in Fig. 8).

If the two-dimensional convex figures Q and Q_1 do not contain a common interior point but intersect in some part, q_0, of their boundaries, then q_0 is either zero-dimensional or one-dimensional; that is, it is a point (Fig. 6) or a line, a half-line, or a line segment (Fig. 8). If one of the figures Q and Q_1 is bounded, then the common part, q_0, of their boundaries may only be either a point or a line segment. In such cases we shall say that Q and Q_1 *adjoin* each other at this point or line segment.

THEOREM 4. *If from a point O inside a bounded convex figure Q a half-line OL is drawn, then this half-line intersects the boundary of Q in one and only one point.*

Proof. Let us draw the half-line OL_1 having direction opposite to the direction of half-line OL (Fig. 9). The two half-lines OL and

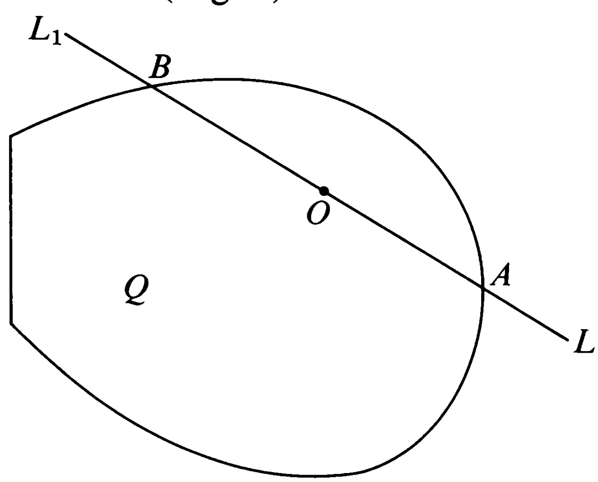


Fig. 9

OL_1 together form straight line LL_1. The intersection of the line LL_1 with the bounded convex figure Q is a bounded convex figure, and so it is some line segment AB. Segment AB contains point O, which is interior to Q. Then all points of this line segment except its end points, A and B, lie inside Q. But of points A and B, one belongs to half-line OL and the other to half-line OL_1. These points lie on the boundary of figure Q. Thus, each of half-lines OL and OL_1 intersects the boundary of figure Q in one and only one point.

In Figure 10*a*, line r divides the convex figure Q into two parts, Q_1 and Q_2, adjoining each other along line segment AB. Both Q_1 and Q_2 are also convex figures, according to the following argument. The line r divides the whole plane into two half-planes R_1 and R_2, each of which is a convex figure. Figure Q is separated into two parts: part Q_1, which is the intersection of two convex figures R_1 and Q, and part Q_2, which is the intersection of two convex figures R_2 and Q. Consequently, Q_1 and Q_2 as intersections of convex figures are convex figures themselves.

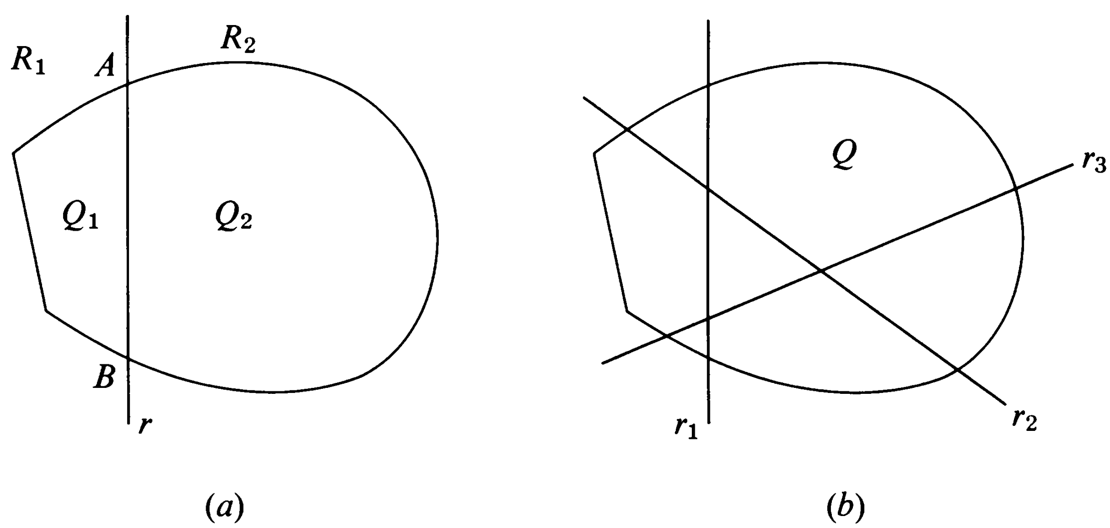


Fig. 10

It is easy to generalize this for any number of lines. Suppose that we are given straight lines $r_1, r_2, \ldots, r_n$ intersecting a convex figure Q (Fig. 10*b*). These lines divide Q into several parts. Each of these parts is a convex figure, since line r_1 can divide Q into only two convex parts, line r_2 can divide each of these two parts in turn into two smaller convex parts, and so forth.

3. SUPPORTING LINES FOR TWO-DIMENSIONAL CONVEX FIGURES

Now suppose that we have a straight line r and a two-dimensional convex figure Q; for simplicity we shall consider Q to be bounded. There are four possible arrangements:

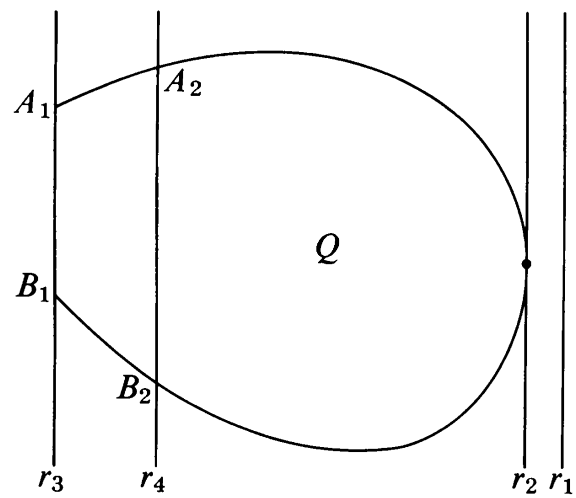
Case 1. r and Q have no points in common (line r_1 in Fig. 11).

Case 2. r and Q have one point in common (line r_2 in Fig. 11).

Case 3. The intersection of r and Q is a line segment belonging entirely to the boundary of Q (segment A_1B_1 of line r_3 in Fig. 11).

Case 4. The intersection of r and Q is a line segment lying entirely (with the exception of its end points) inside Q (segment A_2B_2 of line r_4 in Fig. 11).

Fig. 11



DEFINITION. *A straight line r is a* supporting line for figure Q *if*

(a) *all of figure Q lies on one side of line r and*

(b) *line r has at least one point in common with the boundary of Q.*[1]

Supporting line r is said to adjoin figure Q in point A *if A is a common point of r and the boundary of Q.*

Thus, in Cases 2 and 3 above, line r is a *supporting line* for Q. Also, a tangent to a circle is a supporting line for the circle, and a line on which a side of a triangle lies is a supporting line for the triangle.

Now let us consider a point P on the boundary of a convex figure Q (Fig. 12) and also every possible half-line from P containing other points of Q besides P. The set of points lying on these half-lines[2] is designated by R. We assert that:

[1] A supporting line may also be defined as a *straight line which contains boundary points but does not contain interior points of figure Q.*

[2] And on the half-lines in the limiting position of these half-lines, for example, PL and PL_1 in Figure 12.

Figure R as defined above is convex.

Proof. Let points A and B lie on half-lines PA and PB, containing, respectively, points A_1 and B_1 of figure Q. Line segment A_1B_1 is composed entirely of points of Q, since Q is convex. Let us join all points of this segment with point P by half-lines. These half-lines lie in R and they fill angle APB, which, thus, lies in R. Line segment AB lies in this angle and consequently lies in R. This proves the convexity of R.

A convex figure composed of half-lines from a point P can only be an angle with vertex at P and not greater than π. (An angle greater than π is obviously not convex.)

Proof. Let us label the angle LPL_1. There are two cases.

Case 1. The angle LPL_1 is equal to π (Fig. 13), and half-lines PL and PL_1, being extensions of each other, merge into one straight line, which is a supporting line for Q. In this case, this line is the *only* supporting line passing through P. Every other line through P passes through some interior points of figure Q.

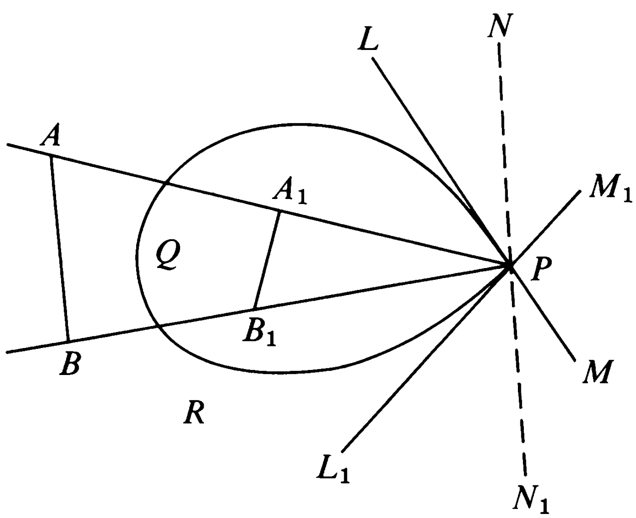


Fig. 12

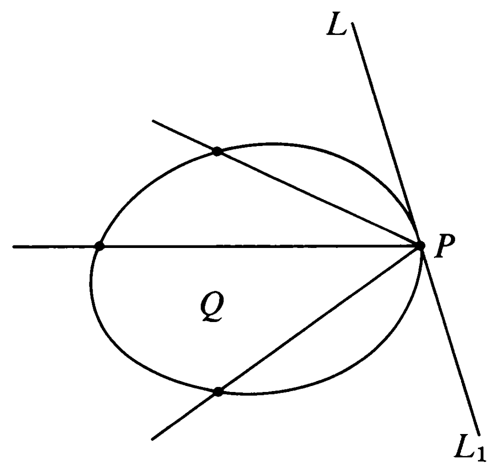


Fig. 13

Case 2. The angle LPL_1 is less than π (Fig. 12). Let us extend the half-lines PL and PL_1 through P to form lines LM and L_1M_1. Since Q is contained in the angle LPL_1, all lines NN_1 passing through P within the vertical angles MPL_1 and M_1PL adjacent to angle LPL_1 also have no points in common with Q except the boundary point P; that is, all such lines are supporting lines. In this case, therefore, the supporting lines for Q at the point P fill out the two vertical angles bounded by the supporting lines LM and L_1M_1, and the point P is called a *singular* (angular) point of the boundary of Q.

4. DIRECTED CONVEX CURVES AND DIRECTED SUPPORTING LINES

Consider a point M moving along the boundary q of a convex figure Q. It is possible to establish two directions of motion of point M along the curve q. The counterclockwise motion is called the *positive* motion around the curve q (Fig. 14), and the motion in the opposite direction is called the *negative* motion around the curve q (Fig. 15).

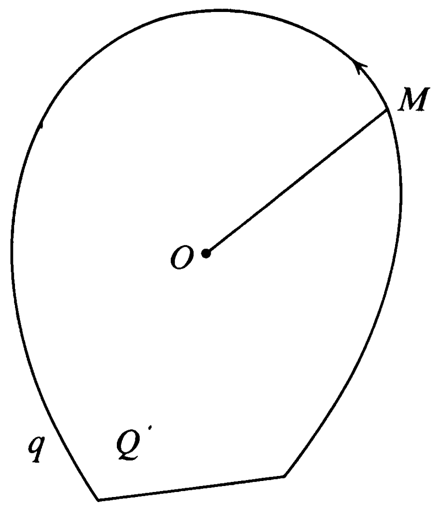


Fig. 14

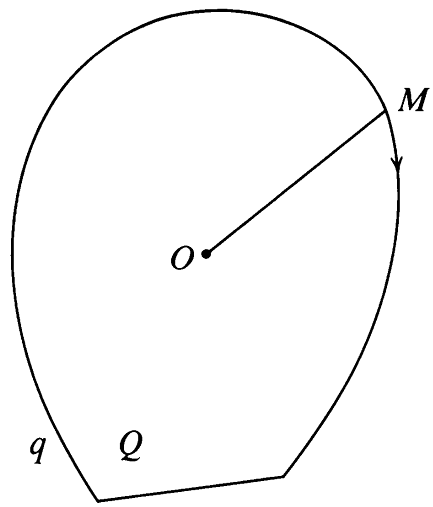


Fig. 15

If we choose a point O inside figure Q and connect point O with the moving point M, the line segment OM changes direction according to the motion of point M. When point M moves in the positive direction around curve q, segment OM rotates counterclockwise around point O. When point M moves in the negative direction, this segment rotates clockwise.

To every straight line it is possible to assign two opposite directions. If one of these is chosen to be the positive direction, then the opposite direction must be considered negative.

Every line r divides the plane into two half-planes. If we look along the line r in its positive direction, then one half-plane will be found on the right side of line r and the other on the left (Fig. 16).

Fig. 16

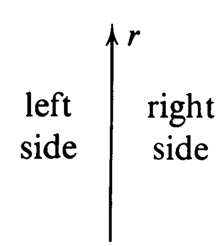
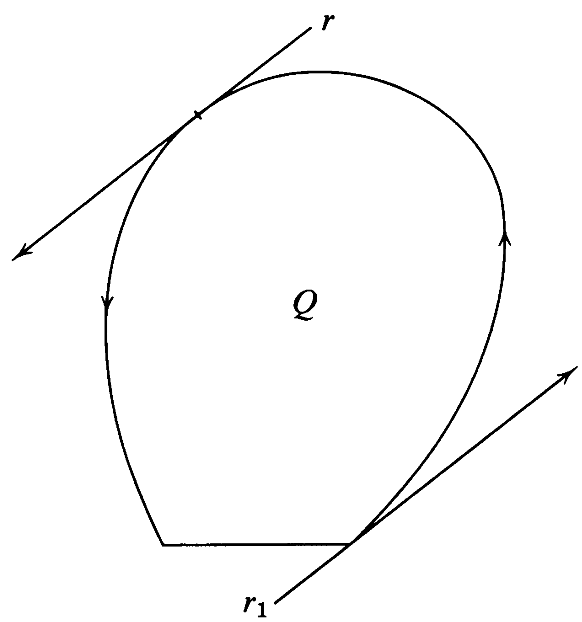


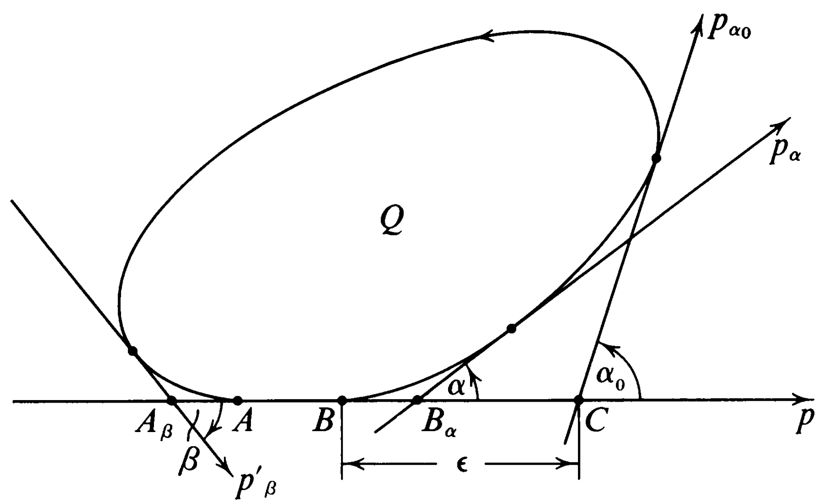
Fig. 17



Now let us consider a convex figure Q and one of its supporting lines r. We shall consider the *positive* direction along r to be that direction on r for which figure Q is found on the left side of r (Fig. 17). By such a definition, the two parallel supporting lines, r and r_1, of the figure Q shown in Figure 17 have opposite directions. Also, a point moving along the boundary (convex curve) in the positive direction and crossing the point of contact of a supporting line r will pass r in its positive direction.

Now let p be a directed supporting line for a convex figure Q (Fig. 18*a*), and let AB be the line segment which the line p has in common with the figure Q. Of course, AB may be a single point ($A = B$). Denote by p_α the supporting line making the positive[1] angle α with p, and label the point of intersection of p and p_α as B_α. For

Fig. 18*a*



[1] The positive angle α is measured counterclockwise from the positive direction of p to the positive direction of p_α.

smaller and smaller angles α, the point B_α approaches closer and closer to the point B. In other words:

Given any positive number ϵ whatsoever, for sufficiently small angles $\alpha > 0$, the length of the line segment BB_α is less than ϵ.

Proof. Let C be the point on the line p at a distance ϵ in the positive direction from the point B. Suppose that p_{α_0} is a supporting line passing through the point C. The line p_{α_0} makes an angle α_0 with the line p. If $0 < \alpha < \alpha_0$, then p_α must intersect p in a point B_α which lies between B and C, so that the distance BB_α is less than ϵ. For, if BB_α were not less than ϵ, then B_α would lie to the right of the point C as in Figure 18*b*. Since $\alpha < \alpha_0$, p_α would meet p_{α_0} at

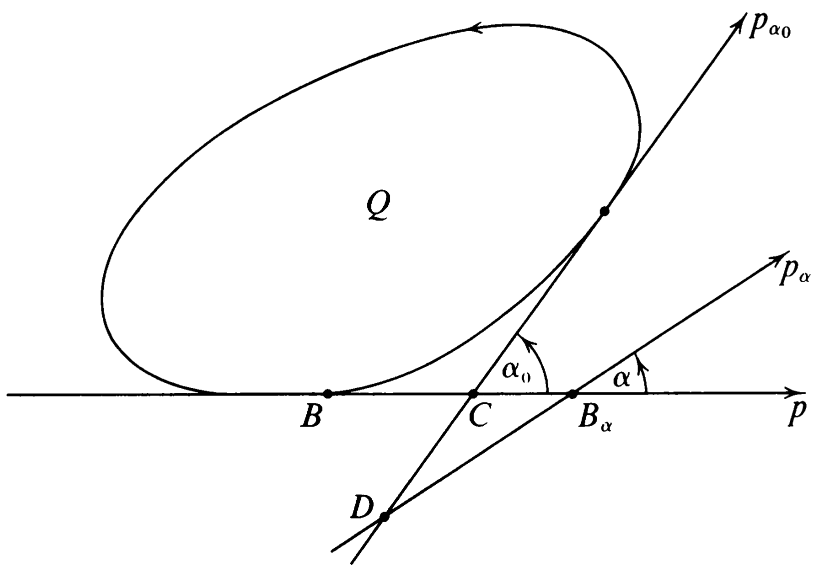


Fig. 18*b*

a point D below p. Then the line p_α would be separated from the figure Q by the lines p and p_{α_0}, which intersect in the point C. Thus, the line p_α would not be a supporting line for the figure Q. This contradicts our assumption that p_α is a supporting line. Therefore, B_α lies between the points B and C, and $BB_\alpha < \epsilon$.

Analogously, if p_β' is a supporting line forming with p a negative angle β (Fig. 18*a*), and A_β is the point of intersection of p and p_β', then for β approaching zero, A_β approaches A.

5. VECTORS; EXTERNAL NORMALS TO PLANE CONVEX FIGURES

DEFINITION. *A* vector *is a line segment which has initial and terminal points and a direction from the initial point to the terminal point.*

The vector with initial point A and terminal point B is denoted by $\overrightarrow{AB}$. Segment AB yields two vectors, $\overrightarrow{AB}$ and $\overrightarrow{BA}$, of opposite directions. Two vectors, $\overrightarrow{AB}$ and $\overrightarrow{CD}$, will be considered to be *parallel vectors* if they are parallel in the ordinary sense and have the same direction (Fig. 19). Vectors, $\overrightarrow{AB}$ and $\overrightarrow{A_1B_1}$, are said to be *equal* is they are parallel and equal in length. The statement

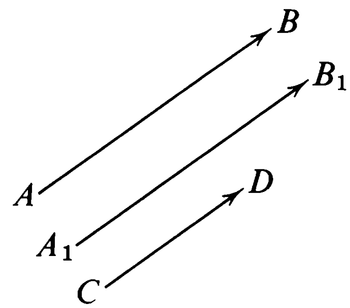


Fig. 19

$$\overrightarrow{AB} = s\overrightarrow{CD}$$

(s a positive number) means that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel and the ratio of their length is s.

Recall the definition of a convex polygon given in section 1. Every side of a convex polygon Q is a part of some supporting line r (Fig. 20). Since there can be only one pair of supporting lines parallel to a given line (see Fig. 11), there cannot be more than two mutually parallel sides of a convex polygon.

Any supporting line of the polygon Q on which no sides of the polygon lie passes through one of its vertices, as lines r_1, r_2, and r_3 in Figure 20. In general, since every side a of a polygon Q is part of a supporting line a' for Q, there is established on a a positive direction, namely that for which polygon Q is situated entirely to the left of the line a'.

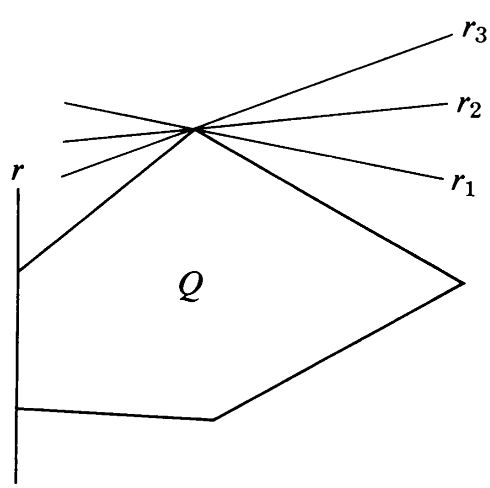


Fig. 20

Definition. *A vector is called an* external normal to polygon Q *at a point A of one of its sides if it is perpendicular to that side and is directed outward from the polygon* (Fig. 21).

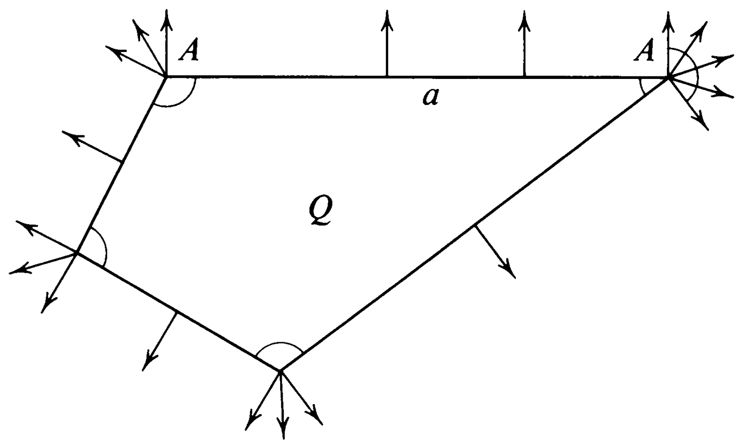


Fig. 21

In every convex polygon there can be only one side with an external normal parallel to a given vector.

More generally, we have:

Definition. *A vector is called an* external normal to a convex figure Q *at a point A if it is perpendicular to a supporting line passing through A and is directed outward from the figure Q* (Fig. 22).

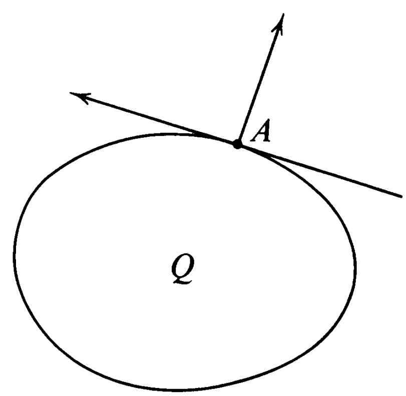


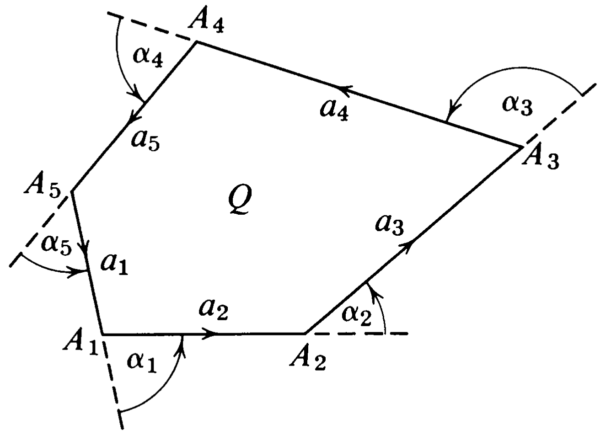
Fig. 22

With this more general definition applied to a convex polygon Q, we see that at every vertex A it is possible to draw an infinite set of external normals, filling an angle supplementary to the vertex angle at A (Fig. 21).

6. CIRCUIT OF A POLYGON; LENGTH OF A CONVEX CURVE

Now let us imagine a man going around the boundary of a convex polygon Q of n sides in the positive direction. During this circuit the interior of the polygon will always be to the man's left. He moves along side a_1 (Fig. 23) to vertex A_1 and then passes onto the adjacent side a_2, turning left through an angle α_1. At each of the vertices the traveler passes from one side of the polygon to the next one, each time turning left through an angle of less than π radians.

Fig. 23



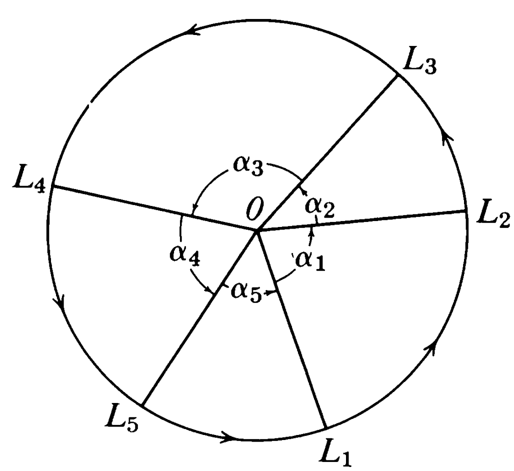
Finally, at vertex A_n he turns left through angle α_n and returns to side a_1. The angles of turning, $\alpha_1, \alpha_2, \ldots, \alpha_n$, are the *exterior angles* of the polygon, and *their sum is equal to four right angles:*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 2\pi.$$

Another way to look at this is to consider that a supporting line at vertex A_1 is rotated through angle α_1 from the direction of side a_1 to the direction of side a_2. At the succeeding vertices, $A_2, A_3, \ldots, A_n$, the supporting line is rotated through angles totaling 2π; that is, the supporting line makes a complete revolution.

Still another way of seeing this is to lay off from a point O vectors, $\overrightarrow{OL_1}, \overrightarrow{OL_2}, \ldots, \overrightarrow{OL_n}$, of unit length parallel to the sides $a_1, a_2, \ldots, a_n$ of the polygon (Fig. 24). Angles $L_1OL_2, L_2OL_3, \ldots, L_nOL_1$ are equal to angles $\alpha_1, \alpha_2, \ldots, \alpha_n$, and the sum of these angles is 2π. The ends, $L_1, L_2, \ldots, L_n$, of the vectors lie on a unit circle. If we move continuously around the circle counterclockwise, starting at point L_1, we pass through all the points $L_1, L_2, \ldots, L_n$ in order, returning to L_1. Thus, we make exactly one trip around the circle.

Fig. 24



An analogous construction can be made for nonconvex polygons. For example, in Figure 25 we have a nonconvex quadrilateral with

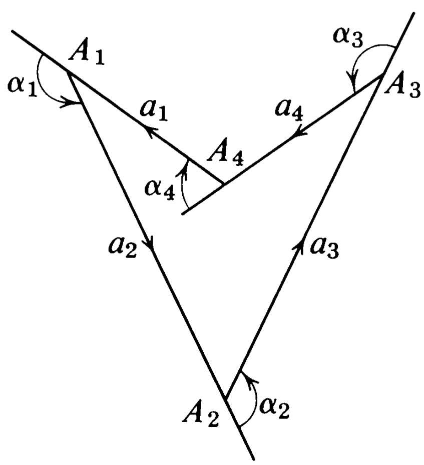


Fig. 25

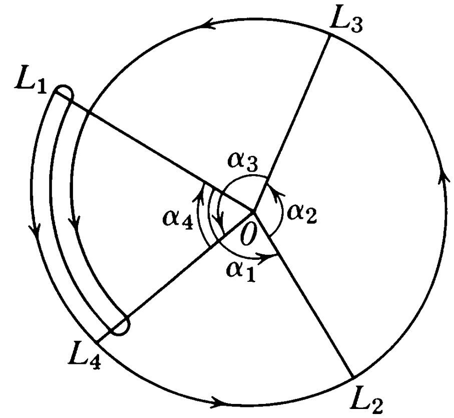


Fig. 26

vertices A_1, A_2, A_3, and A_4. Corresponding to these vertices are the four points L_1, L_2, L_3, and L_4 on the circle with radius 1 and center at O (Fig. 26). Vectors $\overrightarrow{OL_1}$, $\overrightarrow{OL_2}$, $\overrightarrow{OL_3}$, and $\overrightarrow{OL_4}$ are parallel, respectively, to sides a_1, a_2, a_3, and a_4. In making a circuit of points L_1, L_2, L_3, and L_4, we first pass along $\widehat{L_1L_2}$, $\widehat{L_2L_3}$, and $\widehat{L_3L_4}$ in the positive direction, but then we must pass along $\widehat{L_4L_1}$ in the negative direction. Thus, we cannot go around points L_1, L_2, L_3, and L_4 in order and return to L_1 in one positive circuit.

In general, if in making the circuit of points $L_1, L_2, \ldots, L_n$, and L_1 in order, we do not make a single trip around the circle or if we make a forward and backward motion along the circle, then the polygon $A_1A_2 \ldots A_n$ is not convex. We shall use this fact later (cf. section 16) as a test to find out whether there exists a convex polygon with a given set of directed sides.

On the boundary, q, of a two-dimensional bounded convex figure, Q, (Fig. 27), let us mark off points $B_1, B_2, \ldots, B_n$ in order. (The points are said to be in *cyclic* order, because B_1 follows B_n.) Connecting these points in order by line segments, we obtain a polygon, $B_1B_2 \ldots B_n$. Such a polygon is said to be *inscribed in the curve* q. This construction separates figure Q into the following parts: the *inscribed polygon* $B_1B_2 \ldots B_n$ and the *segments* shaded in Figure 27. All these parts are convex figures; in particular, the inscribed polygon is a convex polygon.

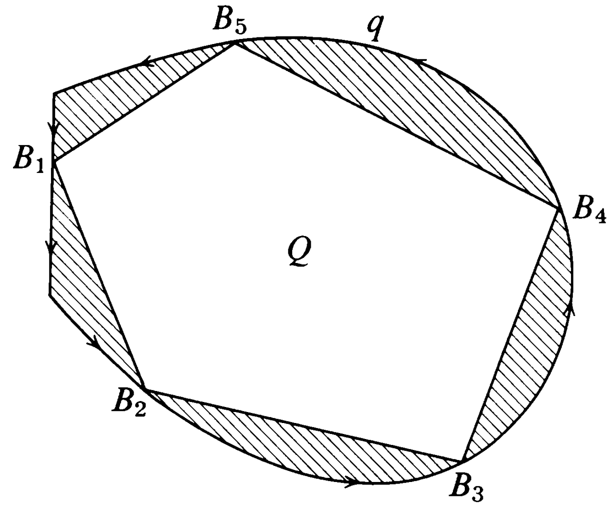


Fig. 27

A convex polygon is said to be *circumscribed about the convex figure Q* if it contains Q while its sides are parts of supporting lines of Q.

Now let us imagine inscribing in the convex curve q a sequence of convex polygons Q_n with n sides, where n increases indefinitely and the length of the greatest side approaches 0. It is possible to prove that *as n increases, the perimeters of the inscribed polygons Q_n approach a limit.* This limit is then taken as the *length of the convex curve.*

It can be proved by elementary geometry that if a convex polygon R_0 lies inside a convex polygon R_1, then the perimeter of R_0 is less than the perimeter of R_1. Then by passing to the limits, it can be shown that if a convex curve s_0 lies inside a convex curve s_1, then the length of s_0 is less than the length of s_1.

7. CONVEX SOLIDS

Next we shall consider convex figures in three-dimensional space. In some convex figures there exist zero-dimensional, one-dimensional, and two-dimensional convex figures side by side but not in the same plane. Such figures are called *three-dimensional convex figures* or *convex solids.*

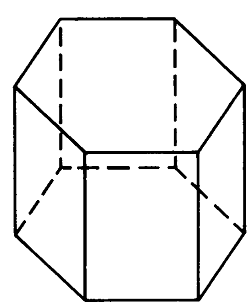


Fig. 28

Examples of convex solids are a sphere, a parallelepiped, and, more generally, a prism with a convex polygon as its base (Fig. 28). A prism with

a nonconvex polygon as its base is an example of a nonconvex solid.

DEFINITION. *A convex solid is called* bounded *if it can be contained inside some sphere; otherwise it is called* unbounded.

A parallelepiped is an example of a bounded convex solid. Examples of unbounded convex solids are a half-space[1] and a circular cylinder (infinitely extended).

The role played by the triangle for the plane is played by the tetrahedron for three-dimensional space (Fig. 29). Let A, B, C, and D be four points not lying in one plane. The tetrahedron $ABCD$ is a triangular pyramid with vertices at these points and four triangular faces, ABC, ABD, ACD, and BCD. We now give a theorem analogous to Theorem 1 of section 1.

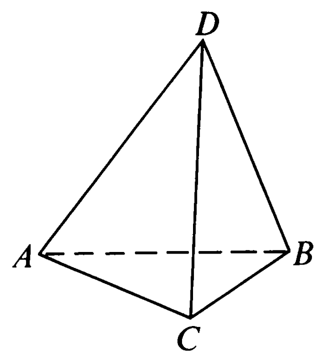


Fig. 29

THEOREM 5. *If a convex solid Q contains four points A, B, C, and D not lying in one plane, then Q contains the whole tetrahedron $ABCD$.*

Proof. By virtue of Theorem 1, section 1, Q contains each of the triangular faces ABC, ABD, ACD, and BCD of the tetrahedron $ABCD$. The tetrahedron contains the line segments AE connecting the vertex A with all points E of face BCD. Since the ends A and E of such segments belong to Q, every such segment AE belongs to Q, and thus the whole tetrahedron belongs to Q.

COROLLARY. *The tetrahedron $ABCD$ is the smallest convex solid containing the points A, B, C, and D.*

If Q is a convex solid, then with respect to Q, all points of space are divided as follows:

(1) Points *exterior to Q*. Around any exterior point it is possible to construct a sphere lying entirely outside of Q.
(2) Points *belonging to Q*. These are divided as follows:
   (*a*) *Interior* points of Q. Around any interior point it is possible to construct a sphere lying entirely inside Q.
   (*b*) *Boundary* points of Q. Every sphere constructed around a boundary point contains both interior and exterior points of Q.

[1] A plane divides space into two *half-spaces*.

Definition. *The set of interior points of* Q *is called the* interior of Q, *and the set of boundary points of* Q *is called the* boundary of Q. *The boundary of a convex solid is a surface and is called a* convex surface.

For example, the boundary of a sphere is its surface; the boundary of a convex polyhedron is the set of its faces.

Theorem 2, section 1, and its corollaries are also true for convex solids.

Theorem 3, section 2, which states that the nonempty intersection of two convex figures is a convex figure, is also true for the three-dimensional case. If Q and Q_1 are convex solids, then the intersection, if it is not empty, is one of the following figures:

(1) A convex solid (three-dimensional convex figure) (Fig. 30)
(2) A two-dimensional convex figure (Fig. 31)
(3) A one-dimensional convex figure (Fig. 32)
(4) A zero-dimensional convex figure (Fig. 33)

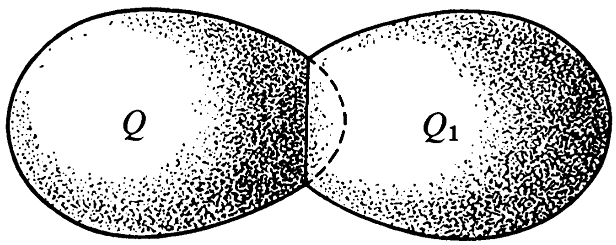


Fig. 30

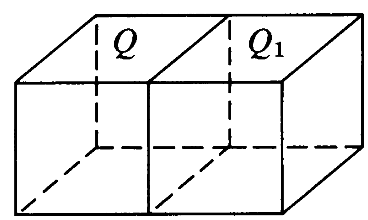


Fig. 31

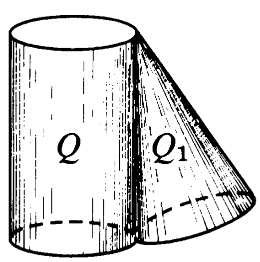


Fig. 32

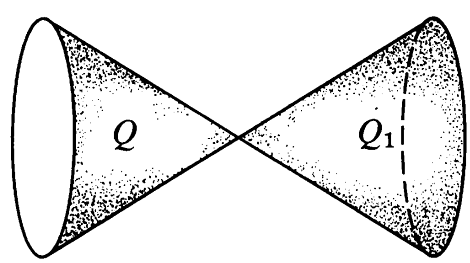


Fig. 33

In situations (2)–(4) we say that the convex solids Q and Q_1 *adjoin* each other.

Theorem 4 of section 2 is also true for convex solids.

8. SUPPORTING PLANES AND EXTERNAL NORMALS FOR CONVEX SOLIDS

Now suppose that we have a convex solid Q and a plane s. There are three possible cases for their arrangement:

Case 1. s and Q have no point in common.

Case 2. s has points in common with the interior of Q.

Case 3. s has at least one point in common with the boundary of Q but none in common with the interior of Q.

DEFINITION. *A plane s is a* supporting plane *for a convex solid Q if s has at least one point in common with the boundary of Q but none in common with the interior of Q.*

Supporting planes play the same role for convex solids in a three-dimensional space as supporting lines do for convex figures in a plane. For example, the supporting planes for a sphere are its tangent planes. Also, a plane in which a face of a convex polyhedron lies is a supporting plane for the polyhedron.

THEOREM 6. *If s is a supporting plane for a convex solid Q, then all of the solid Q lies on one side of the plane s.*[1]

Proof. Let A be an interior point of the solid Q. Then A does not lie on the supporting plane (by virtue of the foregoing definition). Hence, A lies on one side of the plane s. We shall show that all of the solid Q lies on the same side of s as follows: Suppose that there exists a point B of the solid Q lying on the other side of the plane s. Segment AB intersects s in some point C, different from A and B (Fig. 34). Since A is an interior point of Q, it follows, by Theorem 2 of section 1, that point C of segment AB is also an interior point of Q. Plane s, therefore, contains the interior point C of the solid Q, and so it cannot be a supporting plane for Q. The arrival at this contradiction proves the theorem.

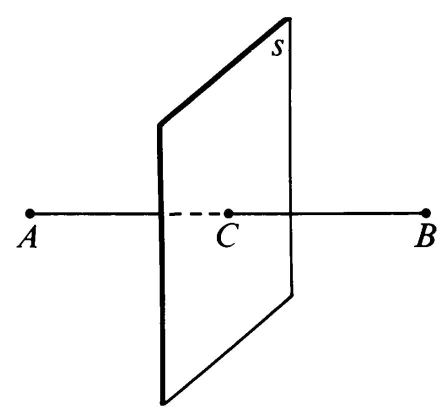


Fig. 34

[1] Sometimes a supporting plane for a convex solid Q is defined as a *plane having a nonempty intersection with Q and such that Q lies on one side of the plane.* Compare these definitions of a supporting plane with the definitions of a supporting line in section 3.

Now let us imagine drawing all possible half-lines OL from an interior point O of a bounded convex solid Q. Each half-line OL intersects the boundary of the convex solid at only one point, by Theorem 4 of section 2. (It was remarked in section 7 that this theorem holds also for convex solids.) We can now prove the following theorems.

THEOREM 7. *For each half-line OL drawn from a point O interior to a convex solid Q there exists a unique supporting plane of Q intersecting this half-line and perpendicular to it.*

Proof. Through point O let us draw a plane s perpendicular to the half-line OL (Fig. 35). The plane s contains interior points of Q.

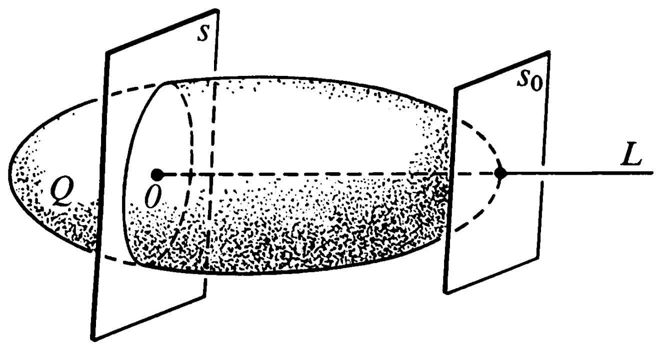


Fig. 35

We now move the plane s in the direction of OL in such a way that it remains perpendicular to OL. There comes a moment when the plane s ceases to contain interior points of Q; this is at position s_0 in Figure 35. Further motion of s in the direction of OL would move it outside Q so that there would be no intersection. Therefore s_0 is the unique supporting plane for Q perpendicular to OL.

THEOREM 8. *For any straight line r and bounded convex solid Q, there exist two supporting planes for Q which are perpendicular to r.*

Proof. There is a line parallel to r passing through an interior point O of the solid Q. The point O divides this line into two half-lines, OL and OL_1. For each of these half-lines there exists one and only one supporting plane perpendicular to the half-line and, hence, perpendicular to r. Consequently, there exists a total of two supporting planes perpendicular to r.

Definition. *A vector is called an* external normal to a polyhedron *Q at a point A of one of its faces if it is perpendicular to this face and is directed outward from the polyhedron* (Fig. 36).

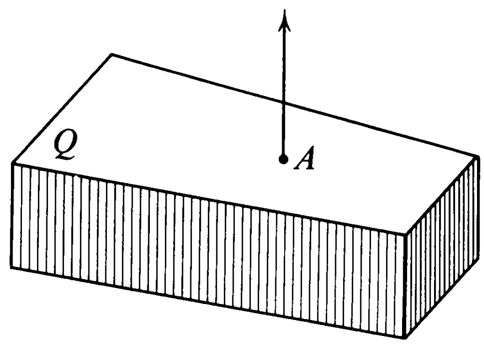


Fig. 36

A convex polyhedron can have only one face with an external normal parallel to a given vector.

Definition. *A vector is called an* external normal to a convex solid *Q at a point A of its boundary if it is perpendicular to a supporting plane passing through A and is directed outward from the figure* (Fig. 37).

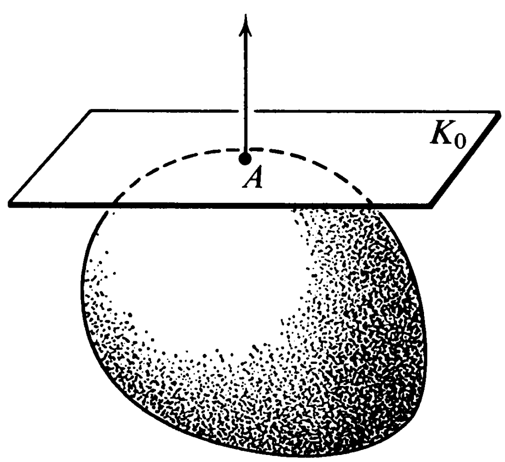


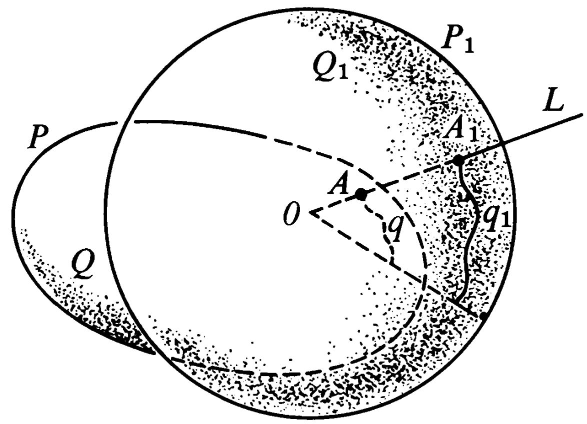
Fig. 37

At each interior point of a face of a convex polyhedron it is possible to draw exactly one external normal with specified length. At each point (other than a vertex) of an edge it is possible to draw an infinite set of normals filling a plane angle supplementary to the corresponding dihedral angle. At each vertex the set of external normals forms a solid angle.

9. CENTRAL PROJECTION; CONES

Consider two convex solids Q and Q_1 with convex surfaces P and P_1 as boundaries. We shall suppose that Q and Q_1 have a common interior point O (Fig. 38). (We can always translate one of the solids Q and Q_1 so that they will have a common interior point.)

Fig. 38



We now imagine drawing all possible half-lines from the point O. Each half-line OL from O intersects the surface P in some point A and the surface P_1 in some point A_1. The points A and A_1 lying on one half-line are said to *correspond* to each other. This correspondence between the points of the surfaces P and P_1 is called a *central projection.*

Now if a curve q is drawn on the surface P, then the points of the surface P_1 corresponding to the points of q make up some curve q_1. We say that the curve q is *mapped* onto the curve q_1 by the central projection. In general, the surface P may be said to be mapped onto P_1.

In particular, for the surface P_1 we may take an arbitrary spherical surface with center at the point O. Thus, *by means of a central projection it is possible to map an arbitrary convex surface onto a spherical surface.*

Note. A central projection is a special case of a *homeomorphism.* A homeomorphism of surfaces (and figures in general) P and P_1 is a transformation of P onto P_1 without folding (that is, distinct points of P go into distinct points of P_1) and without tearing (that is, neighboring points of P go into neighboring points of P_1). A formal definition of homeomorphism is given in Chapter 6.

Half-lines drawn from a point O as for a central projection are said to form a *cone.* Cones play a role in three-dimensional space

comparable to that played in the plane by angles not greater than π radians.

DEFINITION. *A* cone *is an unbounded convex solid made up of half-lines OA from one point, O. Point O is called the* vertex *of the cone.*

Alternatively, we may define a cone with vertex O as a solid, different from the whole space, made up of half-lines from O that contains, together with any pair of half-lines OA and OB forming an angle AOB less than π radians, the entire angle AOB.

Examples of cones are a circular cone (Fig. 39), a convex polyhedral angle, a dihedral angle less than π radians (Fig. 40), and a half-space (bounded by a plane).

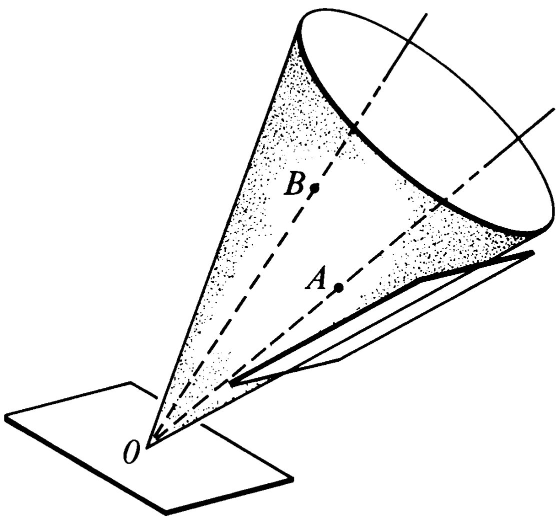


Fig. 39

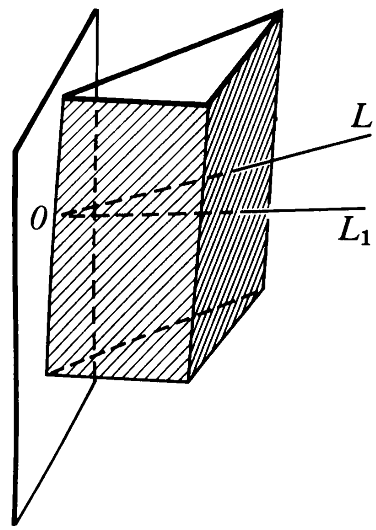


Fig. 40

DEFINITION. *A cone is called a* proper cone *if it contains no entire straight line.*

For example, a circular cone is a proper cone, but a half-space and a dihedral angle are not proper cones. *Every plane angle lying in a proper cone with its vertex at the vertex of the cone is less than* π.

The boundary of a cone is called a *conical surface.* It is composed of half-lines from the vertex of the cone, called the *generators* of the cone.

A supporting plane for a proper cone either borders it along a generator or passes through the vertex O and has no other point in common with the cone (Fig. 39). A supporting plane for a dihedral angle less than π passes through its edge (Fig. 40). The unique supporting plane of a half-space coincides with its bounding plane.

Now let Q be an arbitrary convex solid and O a point of its boundary. Draw all half-lines connecting O with points of Q, and also their limiting half-lines. We obtain some cone K with vertex at O.

DEFINITION. *The boundary K_0 of cone K (defined above) is called the* tangent conical surface to Q at point O.

All supporting planes for K_0 at the point O are supporting planes for Q.

There are three possible cases.

Case 1. The cone K is a half-space, and its boundary K_0 is a plane. This plane, the unique supporting plane for Q at O, passes through the point O of its boundary. We say that the plane K_0 is *tangent* to the convex solid Q at the point O (Fig. 41). Case 1 occurs, for example, at every point on the surface of a sphere.

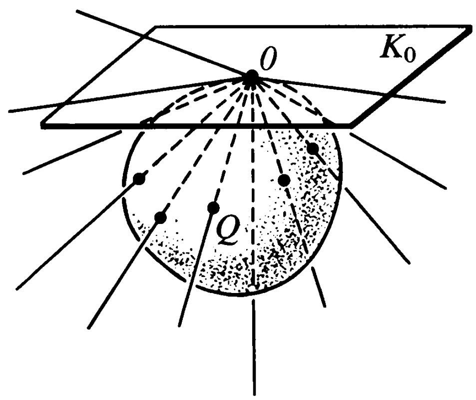


Fig. 41

Case 2. The cone K is a dihedral angle smaller than π. Its boundary K_0 is a pair of half-planes intersecting in the straight line p passing through O (Fig. 42). Then Q has an infinite set of supporting planes passing through the point O, all containing the line p.

For example, Figure 42 shows a finite circular cylinder Q. Its boundary consists of part of a cylindrical surface q_0 and two disks, q_1 and q_2. The surface q_0 meets the disks q_1 and q_2 in the circles r_1 and r_2. Case 2 holds at all points of r_1 and r_2. If O is a point of r_1, then the cone K reduces to a right dihedral angle bounded by the plane K_0' containing q_1, and the

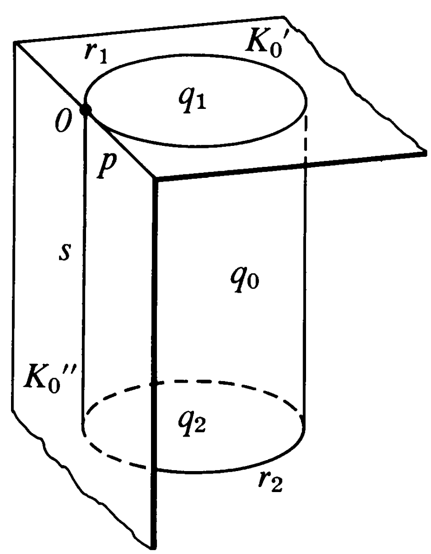


Fig. 42

plane K_0'', tangent to the cylindrical surface along the generator s, passing through O. An infinite set of supporting planes for Q passes through the edge p of this dihedral angle.

Case 3. The cone K is a proper cone. Its surface (a conical surface) is tangent to Q (Fig. 43). In this case, O is called a *singular point* (compare section 3). An infinite set of supporting planes passes through point O.

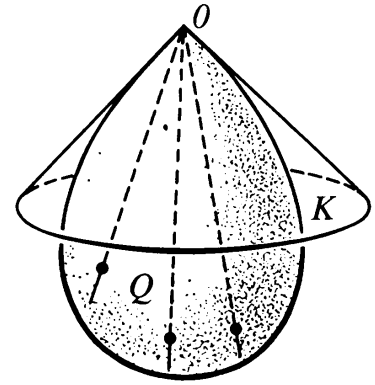


Fig. 43

For example, in a convex polyhedron a vertex is a singular point, Case 2 holds at all points of an edge different from a vertex, and Case 1 holds at all remaining points (interior points of faces). The vertex of a proper cone is also a singular point.

In general, there is either one (Case 1) or an infinite set (Cases 2 and 3) of supporting planes for a convex solid Q passing through any point of its boundary.

10. CONVEX SPHERICAL FIGURES

Let S denote a spherical surface with center at O (Fig. 44). Let A and B be any two points of S not diametrically opposite each other. There is a unique *great circle*[1] on S passing through these two points. We shall denote the smaller of the two arcs of this circle with ends at A and B by $\overset{\frown}{AB}$ and call it the *spherical arc* connecting A and B. Any two points A and B not diametrically opposite each other may be connected by a unique spherical arc, $\overset{\frown}{AB}$.

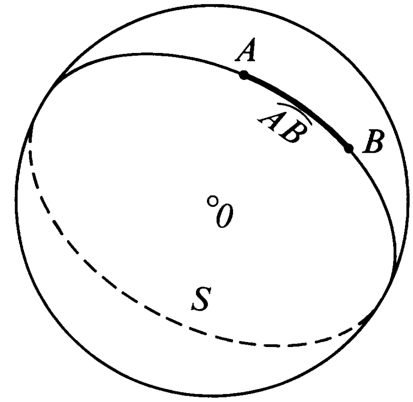


Fig. 44

[1] A circle on the surface of a sphere having the same diameter as the sphere; its center is the center of the sphere.

Definition. *A* spherical convex figure *is a part of a spherical surface S which does not contain a pair of diametrically opposed points of S and which, together with any pair of its points, contains the spherical arc connecting them.*

The region bounded by a *small circle*[1] of the sphere is an example of a spherical convex figure. A point is called an *interior point* of a spherical figure Q if Q contains a small circle with that point as its center. The set of interior points is called the *interior* of the figure. *A spherical convex figure without interior points is a spherical arc.*

Exterior points, *boundary* points, and the *boundary* of a spherical convex figure are defined analogously to those for a plane convex figure (see section 1).

If Q is a spherical convex figure with interior points on the spherical surface S with center O, and if O is connected to all points of Q by half-lines, then these half-lines form a proper cone K, and Q is the intersection of the spherical surface S with this cone K (Fig. 45).

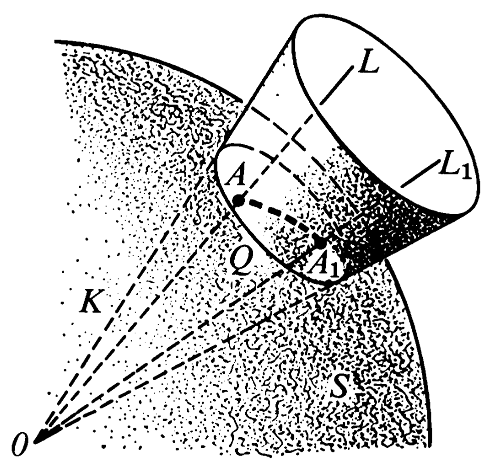


Fig. 45

Proof. Let OL and OL_1 be two half-lines of K intersecting the spherical surface S at points A and A_1 of figure Q. The angle between half-lines OL and OL_1 is the central angle of the arc $\widehat{AA_1}$ of a great circle. Since $\widehat{AA_1}$ belongs entirely to Q and is less than π, the plane angle between OL and OL_1 belongs entirely to K and is less than π, and K is a proper cone.

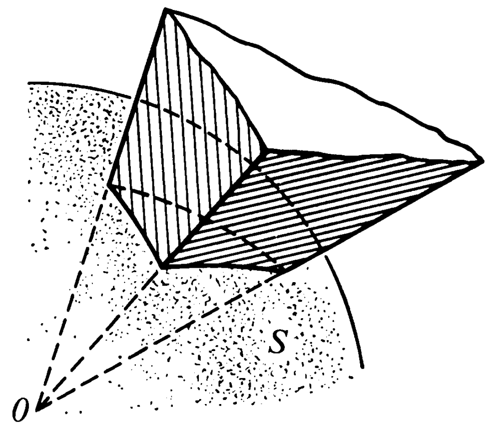
Conversely, *every proper cone with vertex O intersects S in a spherical convex figure.*

[1] A circle on the surface of a sphere that is not a great circle.

Definition. *A* convex spherical *n*-gon *is a convex spherical figure bounded by n spherical arcs.*

A convex spherical *n*-gon is cut out of the surface S by a convex *n*-faced angle with vertex at O. The simplest of these is the *spherical triangle,* which is a convex spherical figure bounded by three spherical arcs. It is cut out of the surface S by a trihedral angle with vertex at O (Fig. 46).

Fig. 46



By arguments analogous to those for Theorem 1, section 1, it is seen that:

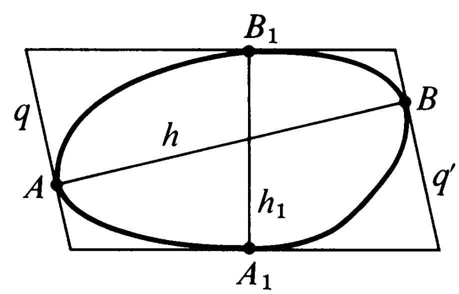
If Q contains points A, B, and C, not all lying on the same spherical arc, then Q must contain spherical triangle ABC.

11. GREATEST AND LEAST WIDTHS OF CONVEX FIGURES

Definition. *The* width *of a bounded plane convex figure (or convex solid) Q in the direction of a line p is the distance between the two supporting lines (or planes) for Q perpendicular to p.*

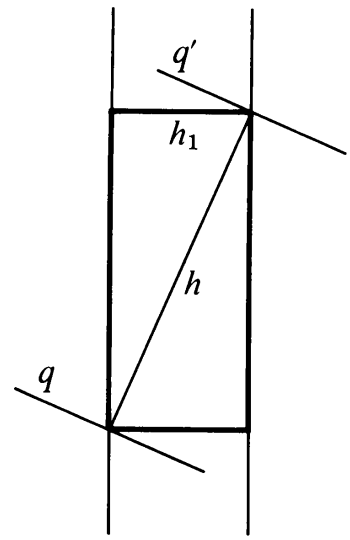
For a bounded convex figure there exists a direction of *greatest width* and a direction of *least width* (Fig. 47). The circle is an example of a figure with constant width in all directions.

Fig. 47



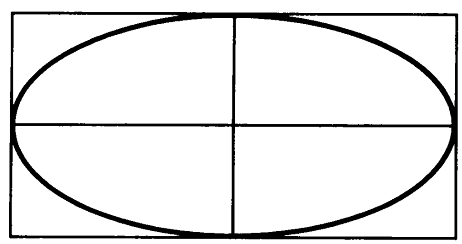
For a rectangle, the directions of the diagonals are the directions of greatest width, and the direction of the shorter side is the direction of least width (Fig. 48).

Fig. 48



For an ellipse, the directions of the major and minor axes are the directions of greatest and least width respectively (Fig. 49).

Fig. 49



Let h be the *greatest width* of a plane figure Q, that is, the greatest of the distances between parallel supporting lines for Q. Let q and q' be any pair of parallel supporting lines the distance between which is equal to h. We shall prove the following properties of the number h and the lines q and q'.

PROPERTY 1. *The distance between any pair of parallel lines s and s' intersecting Q is not greater than h.*

Proof. Let us draw both supporting lines r and r' parallel to s and s'. All of figure Q is contained between r and r'. Consequently, s and s', intersecting Q, are located between r and r'. The distance between r and r' is either greater than the distance between s and s', or in the limiting case is equal to this distance (if the pair r and r' coincides with the pair s and s'). But the distance between r and r' is not greater than h; so we have shown that the distance between s and s' is not greater than h.

Property 2. *The distance between any pair of points A and B of Q is not greater than h.*

Proof. Let us draw a pair of lines s and s' through the points A and B and perpendicular to segment AB. The distance between the points A and B is equal to the distance between lines s and s'. Consequently, by the preceding property, it is not greater than h.

Property 3. Consider a pair of parallel supporting lines q and q' with the distance between them equal to h. Let A and B be points common to Q and q, Q and q', respectively. *Then AB is perpendicular to both q and q'.*

Proof. The distance between points A and B, placed on two parallel lines q and q', is not less than the distance h between these lines. On the other hand, by virtue of Property 2, the distance between A and B is not greater than h. So we know that the distance between A and B is equal to h. But this can happen only in case AB is a common perpendicular to q and q'.

Property 4. *Each of the supporting lines q and q' has only one point in common with Q.*

Proof. Let q, for example, have at least two points A and A' in common with Q. If B is a common point of Q and q', then by the preceding property segments AB and $A'B$ are simultaneously perpendicular to q'. Therefore, we have two perpendiculars from one point B to one straight line q. This contradiction proves our proposition.

We call the greatest distance between points of the figure Q the *diameter* of Q.

Property 5. *The diameter of the figure Q is exactly equal to h.*

Proof. The distance between any two points of the figure Q is not greater than h. On the other hand, the distance between its points A and B (Property 3) is equal to h.

Property 6. *If A and B are two points of Q with distance h (the diameter of figure Q) between them, then A and B are common points of Q and a pair of supporting lines q and q' perpendicular to segment AB.*

Proof. Let us draw two lines r and r' through the points A and B, and perpendicular to AB. The distance between these lines is equal to h. Draw a pair of supporting lines q and q' for Q parallel to r and r'. If the pair q and q' does not coincide with the pair r and r', then the distance between q and q' is greater than h. But this contradicts the definition of h. Therefore, $r = q$ and $r' = q'$ are supporting lines for Q.

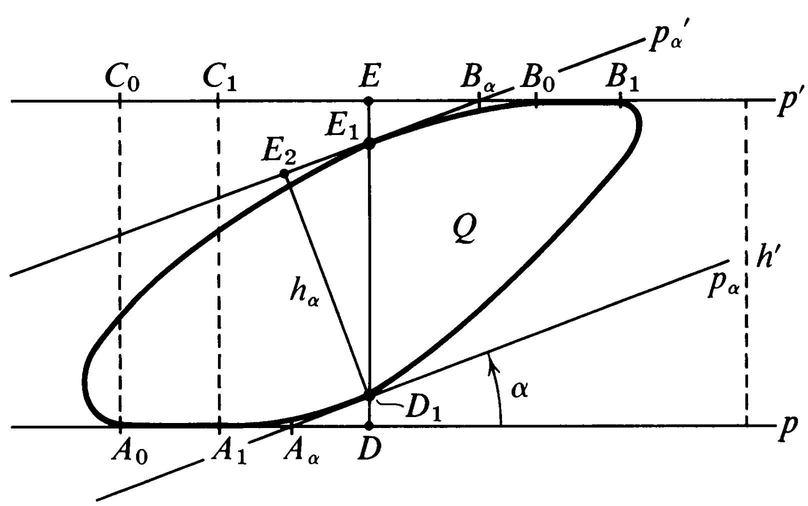
For example, the diameter of a parallelogram equals the length of the greater of its diagonals. The most widely separated pair of supporting lines is the pair of straight lines perpendicular to this diagonal and passing through its end points (in the case of a rectangle there are two such pairs).

For the circle the diameter is the usual diameter. The distance between any pair of parallel supporting lines (tangents) is equal to the diameter.

The diameter of a triangle is the length of the longest of its sides. The most widely separated pair of its supporting lines is a pair of straight lines perpendicular to the longest side of the triangle and passing through its endpoints. (If the triangle is equilateral, there are three such pairs.)

Now let h' be the *least width* of a convex figure Q, and p and p' be a pair of supporting lines separated by a distance exactly equal to h'. Let A_0A_1 and B_0B_1 be the line segments along which p and p' respectively border Q,[1] C_0 and C_1 be the projections of A_0 and A_1 on p' (Fig. 50).

Fig. 50



[1] It is possible that point A_0 may coincide with A_1, or B_0 with B_1, or simultaneously A_0 with A_1 and B_0 with B_1.

We shall show:

The segments C_0C_1 and B_0B_1 have points in common.

Proof. Assume the opposite: let segment B_0B_1 lie entirely outside C_0C_1 (as shown in Fig. 50). Then it is possible to draw a common perpendicular DE to the lines p and p' such that segments A_0A_1 and B_0B_1 will lie on opposite sides of DE. The length of DE is h'.

Let p_α and p_α' be supporting lines making an angle α with p and p',[1] and let A_α and B_α be the points of intersection of p and p_α, and p' and p_α', respectively. As $\alpha \to 0$ (from the discussion in section 4), A_α approaches A_1 and B_α approaches B_0; for sufficiently small α, A_α and B_α, just as A_0 and B_0, lie on different sides of DE. Therefore, p_α and p_α' intersect DE in points D_1 and E_1 such that

$$D_1E_1 < DE = h'. \tag{1}$$

Let D_1E_2 be the perpendicular dropped from D_1 to p_α', and h_α be its length. We have

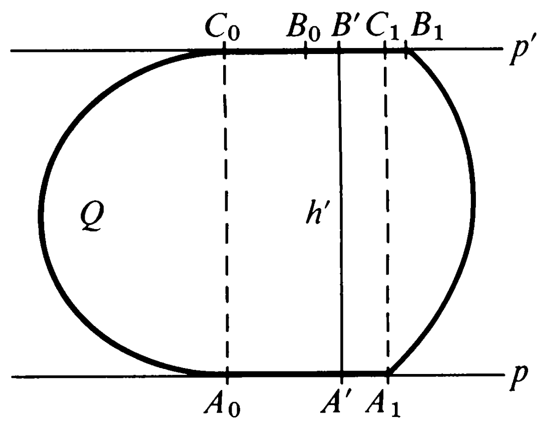
$$h_\alpha < D_1E_1.$$

From this and (1) it follows that

$$h_\alpha < h';$$

that is, the width of Q in the direction of D_1E_2 is less than h', which contradicts the definition of h' as the least width. Therefore, C_0C_1 and B_0B_1 have points in common (Fig. 51).

Fig. 51



Let B' be a common point of C_0C_1 and B_0B_1. The common perpendicular $A'B'$ to the straight lines p and p' has length h'. Point A' is common to Q and p; B' is common to Q and p'. Thus *there exists a line segment $A'B'$ of length h', connecting points of the boundary of Q, perpendicular to supporting lines containing its end points, and having as its direction the direction of least width of Q.*

[1] If A_1 lies to the left of B_0 let α be positive, as in Figure 50. If B_1 lies to the left of C_0, let α be negative.

Combining this result with the result obtained earlier for the line segment of length h (the greatest width), we have:

There exist two line segments $A'B'$ and AB connecting points of the boundary of Q, the directions of which are the directions of least and greatest width, perpendicular to the supporting lines drawn through their end points.

Any bounded three-dimensional convex solid Q has at least three line segments connecting points of the boundary of Q and perpendicular to the supporting planes drawn through their end points. One segment, A_1B_1, has the direction of greatest width. (Its length is equal to the greatest width h_1.) A second segment, A_2B_2, has the direction of least width. (Its length is equal to the least width h_2.) And the third segment, A_3B_3, has intermediate length h_3.

For example, in an ellipsoid, these three segments coincide with the three axes of the ellipsoid.

Note. The length h_3 of the third segment A_3B_3 is determined by the following construction. It is possible to move continuously a pair of parallel supporting planes s and s' of Q, so that they always remain parallel supporting planes, until the plane s comes to the original position of s' and conversely. Such a motion is called a *rotation* of the pair of supporting planes s and s'. We denote this rotation by $\mathfrak{m}$. During a rotation, the distance between the parallel supporting planes will vary, and there will be a position in which this distance (width) assumes its maximum value. This maximum distance depends on the rotation $\mathfrak{m}$. Let $h(\mathfrak{m})$ denote the value of this greatest distance occurring in the rotation $\mathfrak{m}$.

Now we define h_3 as the least of the values $h(\mathfrak{m})$ for all possible rotations $\mathfrak{m}$. There exists a line segment A_3B_3 perpendicular to the supporting planes passing through points A_3 and B_3, for some points A_3 and B_3 of the boundary of Q.[1] The number h_3 is called the *minimax* value of the width of the convex solid. It is included between the maximum and minimum values of the width (between h_1 and h_2). Therefore, for any bounded convex solid there exist three directions, corresponding to the maximum, minimum, and minimax values of the width, and three segments, A_1B_1, A_2B_2, A_3B_3, with these directions and perpendicular to the supporting planes passing through their end points.

[1] Segment A_3B_3 is the segment of greatest width for some rotation $\mathfrak{m}_0$

In n-dimensional space, every bounded convex solid has n segments connecting points of its boundary and perpendicular to the supporting planes passing through their end points.

12. OVALS OF CONSTANT WIDTH; BARBIER'S THEOREM

DEFINITION. *A two-dimensional convex figure for which the distance between any pair of parallel supporting lines (that is, the width in any direction) is constant is called an* oval of constant width.

The circle is the simplest example of an oval of constant width. We shall give other examples.

From the vertices A, B, and C of an equilateral triangle ABC, draw arcs, $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{AC}$, with radii equal to the sides $AB = AC = BC$. Each of these arcs is equal to one sixth of a circle (Fig. 52). *The figure we obtain has constant width.* Of any pair of parallel supporting lines q and q' for this figure, one line passes through one of the vertices A, B, or C, and the other is tangent to $\overset{\frown}{BC}$, $\overset{\frown}{AC}$, or $\overset{\frown}{AB}$ opposite this vertex. The distance between q and q' is equal to the radius of this arc, that is, the length of a side of our equilateral triangle, and is the same for all pairs q and q'.

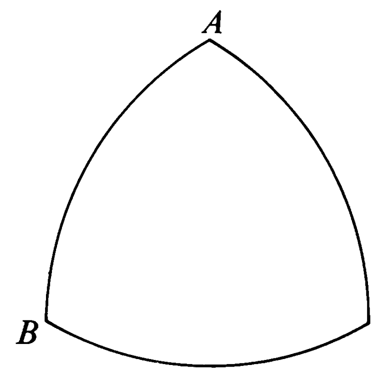


Fig. 52

Let Q be an oval of constant width, and let h be the distance between any two parallel supporting lines for Q. Thus, the "greatest" of these distances coincides with h. Therefore, Properties 1–6, section 11, hold for every pair of parallel supporting lines q and q' of Q. For example, for every pair of supporting lines q and q' for Q, the points, A and B, of q and q' in common with Q lie on a common perpendicular AB to q and q' (Property 3). Each supporting line q has only one point in common with Q (Property 4). The diameter of Q is exactly equal to h (Property 5).

The following theorem gives a remarkable property of convex figures of constant width.

BARBIER'S THEOREM. *The boundary of a convex figure of constant width h has length equal to πh; that is, all figures with the same constant width have boundaries of equal length, which is equal to the circumference of a circle with diameter h.*

For example, the curvilinear triangle ABC in Figure 52 has a boundary composed of three circular arcs of radius h, each of which is a sixth of a circle. Consequently, the length of its boundary is equal to $3 \cdot \frac{1}{6} \cdot 2\pi h = \pi h$.

The proof of the theorem will be based on the definition of the length of a convex curve (the boundary of a convex figure Q; see sections 1 and 6).

Proof. We first consider a regular n-gon C_n circumscribed around a circle of diameter h and a regular n-gon c_n inscribed in the same circle. The length of each side of C_n is equal to $h \tan \frac{\pi}{n}$; each side of c_n equals $h \sin \frac{\pi}{n}$. The perimeters of C_n and c_n are equal to $nh \tan \frac{\pi}{n}$ and $nh \sin \frac{\pi}{n}$, respectively. The circumference, πh, of the circle falls between these lengths, that is,

$$nh \tan \frac{\pi}{n} > \pi h > nh \sin \frac{\pi}{n}, \tag{1}$$

and is equal to the limits of both as $n \to \infty$.

Next we construct around the circle a circumscribed regular polygon C_{2n} with twice as many sides. It has n pairs of parallel sides. We establish a direction of rotation for the circle and for C_{2n}.

Now we consider a convex figure Q and draw around it $2n$ supporting lines parallel to the $2n$ sides of C_{2n} and having the same directions. These supporting lines form a circumscribed $2n$-gon, Q_{2n}, around Q, with vertices $A_1, \ldots, A_{2n}$. (Some sides of this polygon may have length 0; that is, some of its vertices, $A_i, A_{i+1}, \ldots$, may coincide.) The regular polygon C_{2n} with $2n$ sides has n pairs of parallel sides. Therefore, the polygon Q_{2n} also has n pairs of parallel sides, A_iA_{i+1} and $A_{i+n}A_{i+n+1}$ (where $A_{2n+1} = A_1$).

If Q is a figure of constant width, then each of the sides A_iA_{i+1} has only one point, B_i, in common with Q (by Property 4 of section 11, applied to figures of constant width). By connecting points $B_1, B_2, \ldots, B_{2n}$, we shall construct a polygon inscribed in Q and denote it by q_{2n}. We also note that B_iB_{i+n} is perpendicular to A_iA_{i+1} and $A_{i+n}A_{i+n+1}$ by Property 3.

The length of the boundary of the convex figure Q has been defined (section 6) as the common limit of the perimeters of the circumscribed polygons, Q_{2n}, and the inscribed polygons, q_{2n}, as n

becomes infinite. We are to show that for a convex figure Q of constant width h this limit exists and is equal to πh. The proof is based on the following lemma.

LEMMA. *The perimeters of q_{2n} and Q_{2n} are contained between the perimeters of c_n and C_n.*

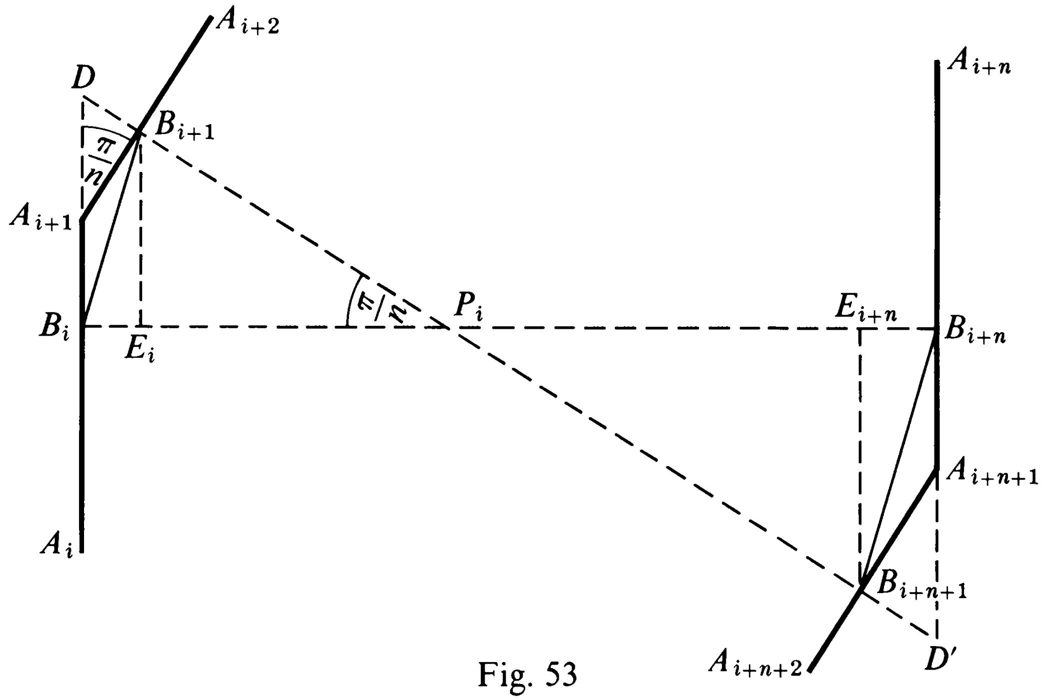


Fig. 53

Proof. Figure 53 shows opposite portions of Q_{2n} and q_{2n}. Consider the part of the boundary of Q_{2n} composed of the segments B_iA_{i+1} and $A_{i+1}B_{i+1}$ of its sides A_iA_{i+1} and $A_{i+1}A_{i+2}$ and the segments $B_{i+n}A_{i+n+1}$ and $A_{i+n+1}B_{i+n+1}$ of the opposite sides $A_{i+n}A_{i+n+1}$ and $A_{i+n+1}A_{i+n+2}$. The segment B_iB_{i+n} is a common perpendicular to the parallel supporting lines for figure Q on which lie sides A_iA_{i+1} and $A_{i+n}A_{i+n+1}$. Similarly, $B_{i+1}B_{i+n+1}$ is a common perpendicular to sides $A_{i+1}A_{i+2}$ and $A_{i+n+1}A_{i+n+2}$.

We next extend side A_iA_{i+1} to intersect the extension of $B_{i+1}B_{i+n+1}$ at a point D. Segment $A_{i+1}D$ is longer than $A_{i+1}B_{i+1}$ because the former is inclined and the latter is perpendicular to line $B_{i+1}B_{i+n+1}$. Consequently,

$$|B_iD| = |B_iA_{i+1}| + |A_{i+1}D| \geq |B_iA_{i+1}| + |A_{i+1}B_{i+1}|. \quad (2)$$

We have equality if the side $A_{i+1}A_{i+2}$ reduces to a point and $A_{i+1} = A_{i+2} = B_{i+1} = D$.

Analogously (Fig. 53),

$$|B_{i+n}D'| \geq |B_{i+n}A_{i+n+1}| + |A_{i+n+1}B_{i+n+1}|. \quad (3)$$

Segments B_iB_{i+n} and $B_{i+1}B_{i+n+1}$ intersect at P_i. The angle $B_iP_iB_{i+1}$ is equal to the exterior angle $DA_{i+1}B_{i+1}$ of Q_{2n}. Moreover, by virtue of the construction (sides of Q_{2n} parallel to sides of C_{2n}), this angle is equal to an exterior angle of regular polygon C_{2n}, that is, equal to $\frac{\pi}{n}$. Thus, we have

$$|B_iD| = |P_iB_i| \tan\frac{\pi}{n}, \quad |B_{i+n}D'| = |P_iB_{i+n}| \tan\frac{\pi}{n}, \tag{4}$$

and from formulas (2), (3), and (4) we obtain:

$$\begin{aligned}|B_iA_{i+1}| + |A_{i+1}B_{i+1}| + |B_{i+n}A_{i+n+1}| + |A_{i+n+1}B_{i+n+1}|&\\ \leq |B_iD| + |B_{i+n}D'| = (|P_iB_i| + |P_iB_{i+n}|) \tan\frac{\pi}{n}&\\ = |B_iB_{i+n}| \tan\frac{\pi}{n} = h \tan\frac{\pi}{n}&\end{aligned}$$

(since $B_iB_{i+n} = h$). Thus, the part of the perimeter of Q_{2n} contained between the pair of line segments B_iB_{i+n} and $B_{i+1}B_{i+n+1}$ does not exceed $h \tan\frac{\pi}{n}$. Consequently, the entire perimeter of Q_{2n} is not greater than $nh \tan\frac{\pi}{n}$, that is, not greater than the perimeter of C_n.

Let us now drop perpendiculars from B_{i+1} and B_{i+n+1} to B_iB_{i+n} and denote the bases of these perpendiculars by E_i and E_{i+n}. In any case, the sides B_iB_{i+1} and $B_{i+n}B_{i+n+1}$ of the inscribed polygon q_{2n} are not shorter than these perpendiculars, so that

$$|B_iB_{i+1}| + |B_{i+n}B_{i+n+1}| \geq |E_iB_{i+1}| + |E_{i+n}B_{i+n+1}|.$$

But

$$|E_iB_{i+1}| = |P_iB_{i+1}| \sin\frac{\pi}{n},$$

$$|E_{i+n}B_{i+n+1}| = |P_iB_{i+n+1}| \sin\frac{\pi}{n}; \tag{5}$$

consequently,

$$\begin{aligned}|B_iB_{i+1}| + |B_{i+n}B_{i+n+1}| &\geq (|P_iB_{i+1}| + |P_iB_{i+n+1}|) \sin\frac{\pi}{n}\\ &= |B_{i+1}B_{i+n+1}| \sin\frac{\pi}{n} = h \sin\frac{\pi}{n}.\end{aligned}$$

Thus, the sum of a pair of opposite sides of inscribed polygon q_{2n} is not less than $h \sin \frac{\pi}{n}$. Hence, the perimeter of the entire polygon, having n such pairs of sides, is not less than $nh \sin \frac{\pi}{n}$, that is, not less than the perimeter of c_n.

Therefore, the perimeter of $C_n \geq$ perimeter of $Q_{2n} \geq$ perimeter of $q_{2n} \geq$ perimeter of c_n, and the lemma is proved.

Proof of the theorem (continued). As $n \to \infty$, the perimeters of C_n and c_n converge to the perimeter πh of the circle. Therefore, the perimeters of the inscribed and circumscribed polygons q_{2n} and Q_{2n}, contained between them, also converge to πh. But the common limit of the perimeters of these polygons is by definition the length of the boundary of figure Q. Thus, the theorem is proved.

Another proof of Barbier's theorem may be found in the book by I. M. Yaglom and V. G. Boltyanskii, *Convex Figures,* translated by Paul J. Kelly and Lewis F. Walton (New York: Holt, Rinehart and Winston, 1961). It also contains material on the three-dimensional analog of ovals of constant width. Also see Hans Rademacher and Otto Toeplitz, *The Enjoyment of Mathematics,* translated by Herbert Zuckerman (Princeton, N. J.: Princeton University Press, 1957).

Remark. Draw a pair of parallel supporting lines p and p' for an oval Q of constant width, and another pair q and q' perpendicular to p and p'. Consider the square with sides lying on p, p', q, q' (Fig. 54). The oval Q may be turned continuously so that it always rests against all four sides of the square.

Also, the properties of a figure which can be thus rotated in an equilateral triangle, a generalization of the properties of ovals of constant width, are considered in the book by Yaglom and Boltyanskii mentioned above.

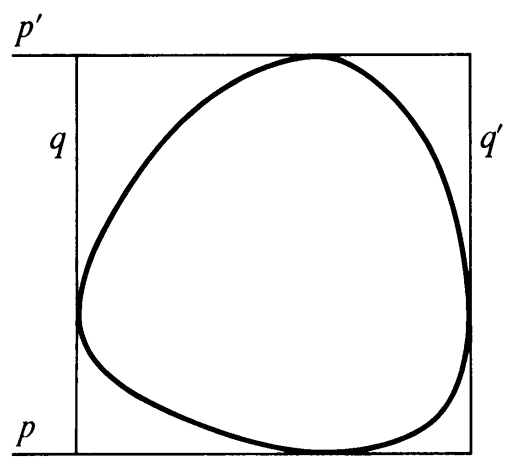


Fig. 54

2. Central-symmetric Convex Figures

13. CENTRAL SYMMETRY AND (PARALLEL) TRANSLATION

DEFINITION. *Two* points *A and B are said to be* symmetric with respect to a point *O if O is the center of the line segment AB.*

Two figures *Q and Q_1 are said to be* symmetric with respect to a point *O if to every point A of the figure Q there corresponds a point B of the figure Q_1 such that A and B are symmetric with respect to point O, and conversely. In this case we also say that Q_1 is obtained from Q by* reflection *in the point O* (Fig. 55).

For example, each side of a pair of opposite sides of a parallelogram is symmetric to the other side with respect to the point of intersection of the diagonals of the parallelogram (Fig. 56).

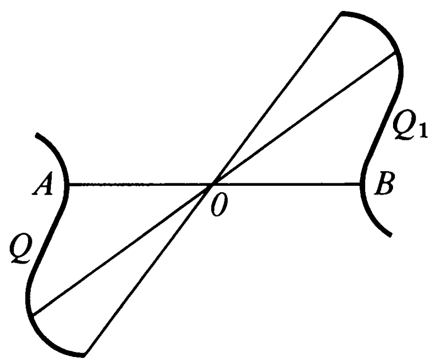


Fig. 55

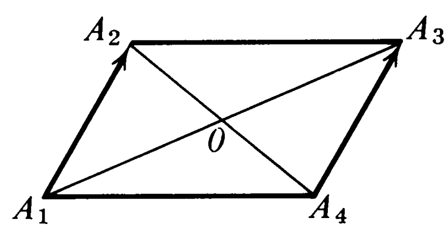


Fig. 56

DEFINITION. *A* (parallel) translation *of a figure Q into a figure Q_1 by a vector, $\overrightarrow{AB}$, is the transformation which sends each point C of Q into a point D of Q_1 where the vector $\overrightarrow{CD}$ is equal to $\overrightarrow{AB}$ and, conversely, each point of D of Q_1 is obtained by moving some point C of Q through the vector $\overrightarrow{CD}$, equal to $\overrightarrow{AB}$* (Fig. 57).

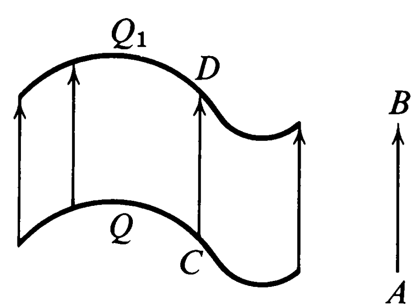


Fig. 57

For example, the side A_2A_3 of the parallelogram $A_1A_2A_3A_4$ is obtained from the parallel translation of the side A_1A_4 by the vector $\overrightarrow{A_1A_2}$ (Fig. 56).

If a point B is symmetric to a point A with respect to a point O and a point B_1 is symmetric to point A with respect to a point O_1, then $\overrightarrow{BB_1} = 2\overrightarrow{OO_1}$ (Fig. 58). For, since segment OO_1 is the segment joining the midpoints of sides AB and AB_1 of triangle BAB_1, it is parallel to segment BB_1 and half as long. Conversely, if a point B is symmetric to a point A with respect to a point O and $\overrightarrow{BB_1} = 2\overrightarrow{OO_1}$, then we can show that B_1 is symmetric to A with respect to O_1.

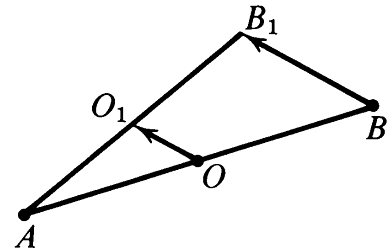


Fig. 58

Reflection of a figure in a point was defined as the reflection of all points of the figure in the same point. Translation of a figure by a vector was defined as the displacement of all points of the figure by the same vector. Hence, from what was proved above we have the following theorem.

THEOREM 1. *If a figure Q_0 is the result of reflecting a figure Q in a point O_0, and Q_1 is the result of reflecting figure Q in a point O_1, then Q_1 may be obtained from Q_0 by translation by a vector equal to* $2\overrightarrow{O_0O_1}$.

If a figure Q_0 is the result of the reflection of a figure Q in a point O_0, and Q_1 is the result of translating figure Q_0 by vector $\overrightarrow{AB}$, then Q_1 may be obtained from Q by reflection in the point O_1, where $\overrightarrow{O_0O_1} = \frac{1}{2}\overrightarrow{AB}$.

DEFINITION. *A figure Q, symmetric to itself with respect to a point O, is said to be a* central-symmetric figure, *and the point O is called its* center of symmetry (Fig. 59).

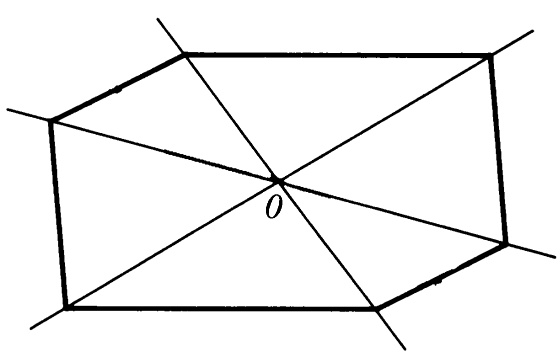


Fig. 59

Examples of central-symmetric figures are the parallelogram (the center of symmetry is the point of intersection of the diagonals), the circle, the ellipse, the parallelepiped (the center of symmetry is the point of intersection of the diagonals), the sphere, the circular cylinder, and so on.

Every side (face) of a central-symmetric polygon (polyhedron) is parallel to a side (face) of equal length (area) with external normal having direction opposite that of the external normals to the given side (face).

The converse of this statement is valid for central-symmetric convex polygons and polyhedra:

THEOREM 2. *If every side (face) of a convex polygon (polyhedron) Q is parallel to a side (face) of equal length (area), then Q has a center of symmetry.*

Proof. For the first case (for polygons) the theorem is easily proved. (The reader is invited to prove it without assistance, using Figure 59.)

The proof for the second case (for polyhedra) is more complicated, but it can proved as a consequence of a theorem of G. Minkowski (1864–1909), which we give in the following form: *If all faces of polyhedra Q_1 and Q_2 with parallel external normals are of equal area, then Q_1 may be obtained from Q_2 by a translation.* The proof of this theorem will be given in Chapter 5 of this book.

Let us reflect a figure Q in a point O, obtaining a figure Q_0. Every two faces a and a_0 of Q and Q_0 with opposite external normals have the same area. But we are given that each face a of the convex polyhedron Q has the same area as the face a' of Q which is parallel to it and has the opposite external normal. In the reflection of Q in point O, face a goes into a face a_0 of Q_0, which has the same area as a and has an external normal parallel to the external normals to a'. Therefore, by the above-mentioned theorem of Minkowski, Q may be obtained from Q_0 by translation by some vector, $\overrightarrow{AB}$.

But by Theorem 1, reflection in point O and translation by vector $\overrightarrow{AB}$ reduce to reflection in a point O_1, where $\overrightarrow{OO_1} = \frac{1}{2}\overrightarrow{AB}$. Hence, Q is obtained from itself by reflection in the point O_1; that is, O_1 is a center of symmetry of Q.

Exercise. Show that any diameter of a central-symmetric convex figure passes through the center of symmetry.

In conclusion we give without proof an interesting theorem of S. P. Olovyanshchikov, a young Leningrad mathematician who was killed in World War II.

OLOVYANSHCHIKOV'S THEOREM. *If all plane cross sections of a convex solid Q dividing its volume in a given ratio $\lambda \neq 1$ are central-symmetric, then Q is an ellipsoid.*

14. PARTITIONING CENTRAL-SYMMETRIC POLYHEDRA

G. Minkowski has proved the following general theorem:

If a convex figure Q (plane or three-dimensional) can be divided into a finite number of central-symmetric parts, then Q is central-symmetric.

Notice that the theorem does not hold for nonconvex figures. In Figure 60, we see a nonconvex hexagon which is not central-symmetric, but which is divided into two central-symmetric parts (parallelograms).

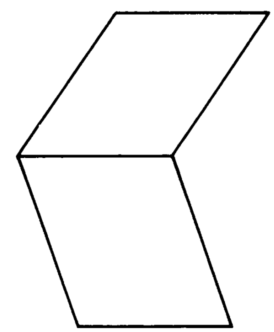


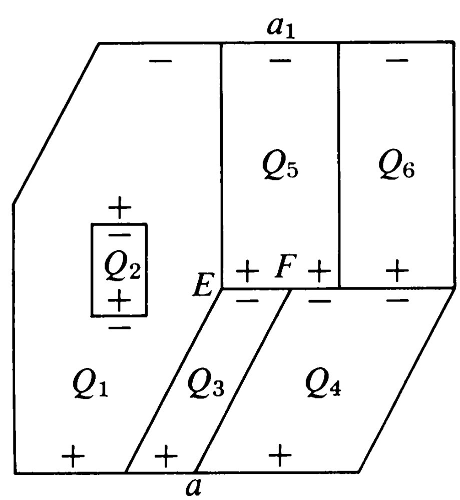
Fig. 60

We shall now prove a special case of the above theorem of Minkowski:

THEOREM 3. *If a convex polygon (polyhedron) Q can be divided into a finite number of central-symmetric polygons (polyhedra), then Q is central-symmetric.*

Proof. We first give the proof for the case of a polygon Q divided into (not necessarily convex) polygons $Q_1, Q_2, \ldots, Q_l$ (Fig. 61). Con-

Fig. 61



sider a side a of this polygon of length d with external normal $\overrightarrow{AB}$ (not shown in the figure). Let the side a_1 of the same convex polygon and parallel to a have length d_1. (If there is no such side, then we shall say that $d_1 = 0$.) We must prove the equality

$$d = d_1. \tag{1}$$

We shall call a side of any of the polygons $Q, Q_1, Q_2, \ldots, Q_l$ a *positive* side if it has an external normal parallel to $\overrightarrow{AB}$, and a *negative* side if it is parallel to a positive side but its external normal has the direction opposite to that of $\overrightarrow{AB}$ (for example, side a_1 of polygon Q). Positive and negative segments shall be segments which are parts of positive and negative sides, respectively.

By the *algebraic* length of a positive or negative segment we mean its ordinary length taken with the sign $+$ or $-$. Thus, the algebraic lengths of sides a and a_1 are denoted by d and $-d_1$ respectively ($d = +d$). Note that the algebraic length is here defined only for sides parallel to a.

In the central-symmetric polygons $Q_1, Q_2, \ldots, Q_l$, each positive (or negative) side has the same length as the corresponding negative (or positive) side (section 13). Therefore, the algebraic sum of the lengths of the sides of any of these polygons is equal to zero. If c denotes the algebraic sum of the lengths of the sides of all these polygons, then

$$c = 0.$$

On the other hand,

$$0 = c = c_1 + c_2 + c_3, \tag{2}$$

where c_1 and c_2 are the algebraic sums of the lengths of all sides of polygons $Q_1, Q_2, \ldots, Q_l$ lying on sides a and a_1, respectively, of the original polygon Q, and c_3 is the algebraic sum of the lengths of all sides of polygons $Q_1, Q_2, \ldots, Q_l$ lying inside Q and parallel to a.

Obviously,

$$c_1 = d, \quad c_2 = -d_1. \tag{3}$$

From (2) and (3) follows:

$$d - d_1 + c_3 = 0. \tag{4}$$

Notice that if two polygons, Q_i and Q_j, border each other along a common segment EF of their sides parallel to a (Fig. 61), then this

segment is a positive segment for one of these polygons and a negative segment for the other. The sides of $Q_1, Q_2, \ldots, Q_l$ parallel to a and lying inside Q are divided into positive and negative segments such that each positive segment corresponds to a negative segment of equal length (in fact, coincides with it), and conversely. The algebraic sum c_3 of all such segments is zero. From this and (4), it follows that

$$d - d_1 = 0,$$

that is, $d = d_1$. Therefore, each side of the convex polygon is equal to the side parallel to it. Then, by Theorem 2, sertion 13, Q has a center of symmetry.

The proof of the theorem for polyhedra proceeds entirely analogously, where we consider faces and parts of faces instead of sides and segments of sides, and areas instead of lengths.

In conclusion we give without proof two theorems. Theorem 4 is due to A. D. Aleksandrov.

THEOREM 4. *If every face of a convex polyhedron Q has a center of symmetry, then the polyhedron Q itself also has a center of symmetry.*

THEOREM 5. *If every face of a convex polyhedron Q has a center of symmetry, then Q can be divided into parallelepipeds.*

15. THE GREATEST CENTRAL-SYMMETRIC CONVEX FIGURE IN A LATTICE OF INTEGERS; MINKOWSKI'S THEOREM

We shall consider systems of straight lines in the plane that divide the plane into strips of equal width. Two such systems of parallel lines divide the plane into congruent parallelograms and form a plane network or *lattice.* The vertices of these parallelograms are called *lattice points,* and the parallelograms themselves are called *the fundamental parallelograms* of the lattice. Lattice points are the points of intersection of the straight lines of the two systems (Fig. 62).

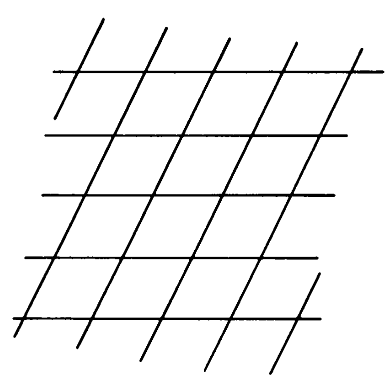


Fig. 62

Analogously, in three-dimensional space we consider systems of parallel planes dividing the space into layers of equal thickness. Three such systems divide the

space into congruent parallelepipeds and form a three-dimensional network or *lattice.* The vertices of these parallelepipeds (in which intersect planes of all three systems) are *lattice points* and the parallelepipeds themselves are the *fundamental parallelepipeds* of the lattice.

DEFINITION. *A lattice as described above in the plane or in three-dimensional space is called a* lattice of integers.

We shall assume that the fundamental parallelograms (parallelepideds) of the lattice have area (volume) equal to 1.

For simplicity, we shall consider a plane lattice of integers for which the fundamental parallelograms are squares (Fig. 63). In this

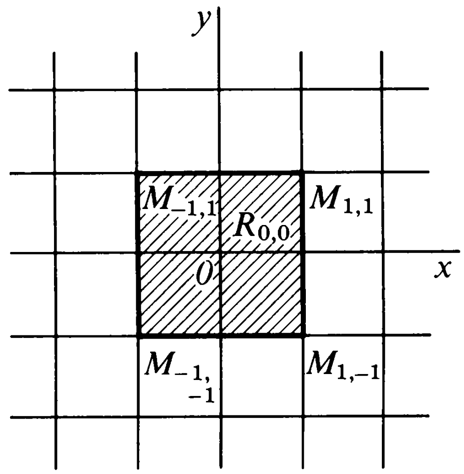


Fig. 63

case the straight lines of the two systems are perpendicular. Choose a lattice point O for the origin. There is a line of each of the two systems passing through this point. Take one of these lines to be the axis of abscissas and the other for the axis of ordinates, and take the length of a side of a fundamental square for the unit of length. Then all lattice points (k, l) have integral (positive, negative, or zero) coordinates k and l. We shall denote a lattice point (k, l) by $M_{k,l}$. If we subject the lattice to an integral translation, that is, translate by a vector $\overrightarrow{OA}$ where $O = M_{0,0}$ (the origin) and $A = M_{p,r}$, then each lattice point $M_{k,l}$ goes into point $M_{k+p,l+r}$, and the entire lattice is carried onto itself.

Analogous definitions are made for a three-dimensional lattice.

Extremal problems are those that concern lines and other figures for which some quantity assumes the least value. They play an important role in mathematics. A simple example of such a problem is to find the shortest curve connecting two points of a plane. Such

a curve, obviously, is a straight line segment. The reader may look through our book *Shortest Paths*[1] for examples of such problems. There is a large selection of interesting extremal problems about convex figures in the book by I. M. Yaglom and V. G. Boltyanskii mentioned at the end of Chapter 1.

We have previously considered the simplest of such problems, that is, finding the smallest convex figure containing three noncollinear points A, B, C. It turned out to be the triangle ABC. In sections 32 and 40 we shall consider the so-called *isoperimetric problem,* that is, the problem of finding the plane figure having the greatest area for a boundary of given length.

We shall now give an interesting extremal problem connected with the arrangement of central-symmetric convex figures in a lattice of integers. A plane convex figure (convex solid) Q is said to *cover* a point P if P is an *interior* point of Q. Suppose that we are given a plane lattice of integers and a convex central-symmetric figure Q with its center a one of the lattice points and not covering any other lattice point. G. Minkowski stated and solved the problem of finding the maximal possible area of such a figure and the analogous problem for a three-dimensional lattice. The following theorem gives the solution.

MINKOWSKI'S THEOREM. (*a*) *The greatest area of a plane central-symmetric convex figure Q whose center of symmetry coincides with one of the lattice points of an integral lattice and which does not cover any other lattice points is equal to* 4.

(*b*) *The greatest volume of a three-dimensional convex central-symmetric solid whose center of symmetry coincides with one of the lattice points of an integral lattice and which does not cover any other lattice point is equal to* 8.[2]

Proof of (*a*). Let $R_{0,0}$ denote the square with vertices at the lattice points $M_{1,1}$, $M_{-1,1}$, $M_{-1,-1}$, $M_{1,-1}$ and center at the point $O = M_{0,0}$ (Fig. 63). The area of this square is equal to 4, and the square $R_{0,0}$ does not cover any lattice points other than its center O. If we translate $R_{0,0}$ by vector $\overrightarrow{OM}_{m,n}$, then it goes over to square $R_{m,n}$ with area also equal to 4 and with center at lattice point $M_{m,n}$

[1] L. A. Lyusternik, *Shortest Paths: Variational Problems* (New York-London· Pergamon Press, 1964).

[2] The theorem has been proved for spaces with n dimensions. The maximal n-dimensional volume under the conditions of the theorem is 2^n (for $n = 2$ and $n = 3$, 2^n equals 4 and 8, respectively).

and vertices at lattice points $M_{m+1,n+1}$, $M_{m-1,n+1}$, $M_{m-1,n-1}$, and $M_{m+1,n-1}$.

Now let $Q_{0,0}$ be a convex central-symmetric figure with center at $O = M_{0,0}$, covering no lattice points other than its center. Let $Q_{m,n}$ denote the figure obtained by translating $Q_{0,0}$ by vector $\overrightarrow{OM}_{m,n}$. We must prove that the area of $Q_{0,0}$ (and, therefore, of each of the figures $Q_{m,n}$) does not exceed 4 (the area of $R_{0,0}$).

We consider the set of squares $R_{2m,2n}$ with centers at points $M_{2m,2n}$ with *even* integral coordinates. The entire plane is thus divided into an infinite set of such squares, which fill the plane and do not overlap each other. Two figures are said to *overlap* each other if they have common interior points. We also consider the set of all figures $Q_{2m,2n}$. We shall now prove the following:

LEMMA. *If $Q_{0,0}$ does not cover any lattice points other than its center, then the figures $Q_{2m,2n}$ do not overlap each other.*

Proof. Suppose that the figures $Q_{2m,2n}$ and $Q_{2m',2n'}$ with centers at the points $M_{2m,2n} = M$ and $M'_{2m',2n'} = M'$ overlap each other, that is, have a common interior point C. We consider the two possible cases:

Case 1. The common interior point C does not lie on the line MM' (Fig. 64).

Fig. 64

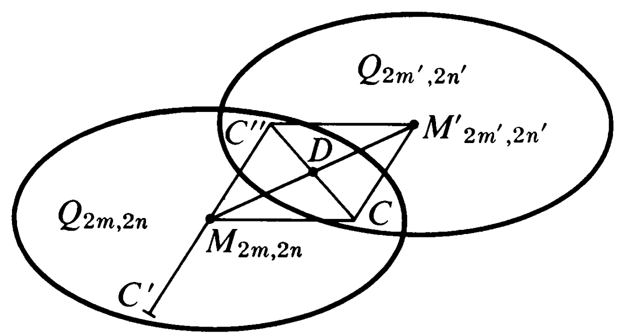


Figure $Q_{2m,2n}$ is obtained from $Q_{2m',2n'}$ by translation by the vector $\overrightarrow{M'M}$; the segment $M'C$ lies entirely inside $Q_{2m',2n'}$ (since M' and C lie inside the convex figure $Q_{2m',2n'}$). If we translate the figure $Q_{2m',2n'}$ by vector $\overrightarrow{M'M}$ so that it coincides with $Q_{2m,2n}$, then the segment $M'C$ goes into an equal and parallel segment MC' lying entirely inside $Q_{2m,2n}$. Let MC'' denote the segment equal and parallel to MC' but lying on the opposite side of point M. This segment is also equal and parallel to $M'C$. Since $Q_{2m,2n}$ has a center of symmetry at point M, segment MC'', as well as segment MC', symmetric to MC'' with respect to point M, lies entirely in $Q_{2m,2n}$.

The quadrilateral $MCM'C''$ has a pair of equal and parallel sides $M'C$ and MC'', so that it is a parallelogram. Its diagonals MM' and CC'' intersect at a point D, their common midpoint. Since the points C and C'' both lie inside the convex figure $Q_{2m,2n}$, the entire segment CC'' lies inside this figure. In particular, the point D lies inside the figure $Q_{2m,2n}$. As the mid-point of the segment MM', point D has coordinates

$$x = \frac{2m + 2m'}{2} = m + m',$$

$$y = \frac{2n + 2n'}{2} = n + n',$$

that is, point D has integral coordinates and consequently is a lattice point. Thus, the figure $Q_{2m,2n}$ covers a lattice point different from its center $M_{2m,2n}$, contrary to the assumption of the lemma.

Case. 2. The common interior point C lies on the segment MM' (Fig. 65).

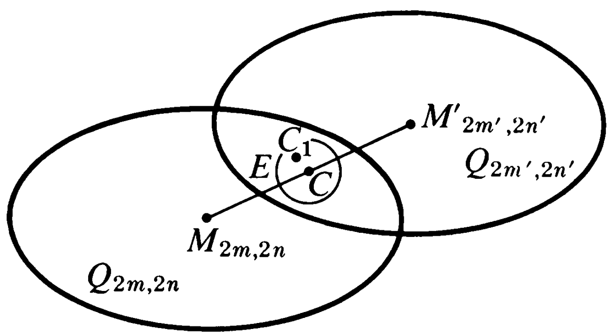


Fig. 65

Then there exists a circle E with center at C which lies entirely inside both figures. This circle contains a point C_1 not lying on the segment MM' but which is an interior point of both figures $Q_{2m,2n}$ and $Q_{2m',2n'}$. We have now reduced this case to the preceding case.

If figures $Q_{2m,2n}$ and $Q_{2m',2n'}$ have a common interior point, then $Q_{2m,2n}$ contains at least one lattice point besides $M_{2m,2n}$, and this contradicts our assumption. Thus, the lemma is proved.

Proof of (*a*) (*continued*). If the figure $Q_{0,0}$ lies entirely inside the square $R_{0,0}$, then its area does not exceed the area of $R_{0,0}$, that is, 4 in this case. But if the figure $Q_{0,0}$ lies only partly in the square $R_{0,0}$ (Fig. 66), then $Q_{0,0}$ is divided into several parts:

Part A_1 is the intersection of $Q_{0,0}$ and $R_{0,0}$, and parts $A_2, A_3, \ldots, A_k$ are the intersections of $Q_{0,0}$ and squares $R_{2m,2n}$ different from

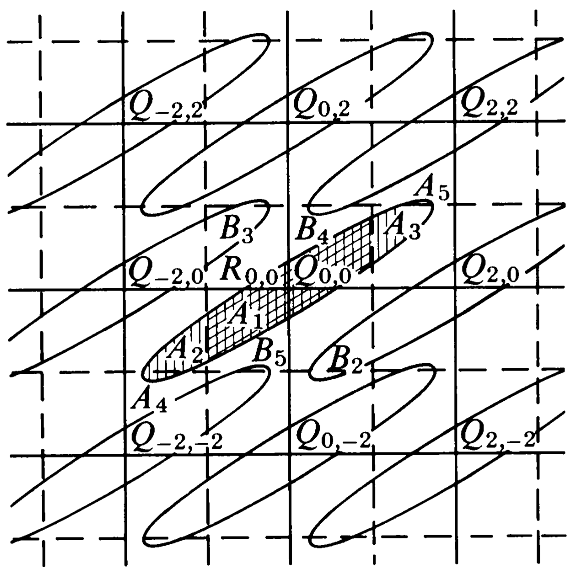


Fig. 66

$R_{0,0}$. Consider one such part A_i of the figure $Q_{0,0}$ ($i = 2, 3, \ldots, k$). It lies in some square $R_{2m,2n}$ with center of symmetry $M = M_{2m,2n}$. The square $R_{-2m,-2n}$ is symmetric to $R_{2m,2n}$ with respect to the origin and has center of symmetry $M' = M_{-2m,-2n}$. Translate the plane by the vector $\overrightarrow{MO} = \overrightarrow{OM'}$. Under this translation the square $R_{2m,2n}$ goes into the square $R_{0,0}$, and the square $R_{0,0}$ into the square $R_{-2m,-2n}$. The figure $Q_{2m,2n}$ with center at $M_{2m,2n}$ goes into the figure $Q_{0,0}$, and $Q_{0,0}$ into $Q_{-2m,-2n}$ with center at M'. Figure A_i, the intersection of $Q_{0,0}$ and $R_{2m,2n}$, goes into figure B_i, the intersection of $Q_{-2m,-2n}$ and $R_{0,0}$.

Thus, in the square $R_{0,0}$ we find a part B_i of the figure $Q_{-2m,-2n}$ congruent to the part A_i of the figure $Q_{0,0}$. Similarly, parts $A_2, A_3, \ldots, A_k$ of figure $Q_{0,0}$ are congruent to parts $B_2, B_3, \ldots, B_k$ of various figures $Q_{2m',2n'}$ (different from $Q_{0,0}$ and from each other). Since the various figures $Q_{2m',2n'}$ do not overlap each other, then their parts B_i lying inside $R_{0,0}$ also do not overlap each other and do not overlap A_1. The figures A_1, B_2, B_3, $\ldots$, B_k cover either the entire square $R_{0,0}$ or a part of it, and do not overlap each other. The sum of their areas does not exceed the area of $R_{0,0}$, that is, 4. But each part A_i of figure $Q_{0,0}$ has the same area as the corresponding B_i. Then the sum of the areas of all the parts $A_1, A_2, A_3, \ldots, A_k$ of figure $Q_{0,0}$ is equal to the sum of the areas of A_1, B_2, $B_3, \ldots, B_k$, and consequently does not exceed 4. The area of $Q_{0,0}$ is equal to the sum of the areas of its parts A_i. Hence, it is not greater than 4. Thus, Minkowski's theorem is proved for the plane case, (a).

In the preceding paragraph we saw that the area of the figure $Q_{0,0}$ is equal to the sum of the areas of the parts $A_1, B_2, B_3, \ldots, B_k$ of the figures $Q_{2m,2n}$ lying in the square $R_{0,0}$.

Because of this, we have the following:

(1) If the area of $Q_{0,0}$ is less than 4, then the parts A_1, B_2, $B_3, \ldots, B_k$ of the figures $Q_{2m,2n}$ do not completely fill the square $R_{0,0}$.

(2) If the area of $Q_{0,0}$ is equal to 4 (that is, equal to the area of $R_{0,0}$ itself), then these parts of the figures $Q_{2m,2n}$ completely fill the square $R_{0,0}$ (without gaps). Also, the parts of the figures $Q_{2m,2n}$ will completely fill any square $R_{2m,2n}$. And since the entire plane is divided into these squares, the figures $Q_{2m,2n}$ cover the entire plane without gaps and without overlapping each other, as was shown above.

We have obtained the following:

(a'). *If a convex figure $Q_{0,0}$ has the maximum area,* 4, *then the figures $Q_{2m,2n}$ fill the entire plane without gaps.* Conversely, *if the convex figures $Q_{2m,2n}$ with centers at the lattice points having even integral coordinates cover the whole plane without gaps and without overlapping each other, then the area of each is 4.*

Thus, the problem of determining the figures $Q_{0,0}$ having maximum area and the problem of determining the figures $Q_{2m,2n}$ covering the plane without gaps and without overlapping are equivalent.

A corresponding development can be given in the three-dimensional case. Let $R_{0,0,0}$ denote a cube with center at the origin, and edges 2 units long and parallel to the coordinates axes. Its vertices lie at the lattice points for which each of the coordinates is equal to 1 or -1. The interior of the cube $R_{0,0,0}$ contains only one lattice point, that is, its center O.

Let us consider for a moment a convex solid $Q_{0,0,0}$ with center at the origin and whose interior contains no lattice point other than its center. By reasoning entirely analogous with that above, we can prove the following statements in order:

1. The preceding lemma for the three-dimensional case: Let $Q_{2l,2m,2n}$ denote one figure with center of symmetry at the lattice point with coordinates $2l$, $2m$, $2n$ obtained from $Q_{0,0,0}$ by translation. *If $Q_{0,0,0}$ does not cover any lattice points other than its center, then the solids $Q_{2l,2m,2n}$ do not overlap each other.*

2. Part (b) of Minkowski's theorem: Inside cube $R_{0,0,0}$ are found part A_1 of solid $Q_{0,0,0}$ and parts $B_2, B_3, \ldots, B_k$ of other solids $Q_{2l,2m,2n}$. The sum of the volumes of these parts $A_1, B_2, B_3, \ldots, B_k$ is equal to the volume of solid $Q_{0,0,0}$. On the other hand, this volume does not exceed the volume of $R_{0,0,0}$, that is, $2^3 = 8$.

3. (b'), corresponding to (a') above: *If solid* $Q_{0,0,0}$ *has the maximal possible volume (equal to 8), then the solids* $Q_{2l,2m,2n}$ *fill the entire three-dimensional space without gaps.* Conversely, *if the solids* $Q_{2l,2m,2n}$ *fill three-dimensional space, then these solids have the greatest possible volume,* 8.

16. FILLING THE PLANE AND SPACE WITH CONVEX FIGURES

We have seen in section 15 that the maximal central-symmetric figures $Q_{2m,2n}$ in an integral lattice fill the plane without gaps. Now we shall determine the form of these figures.

Since the figures $Q_{2m,2n}$ do not overlap each other, and, on the other hand, do not leave any gaps, they must adjoin each other, that is, have common parts of their boundaries. Since a common part of the boundaries of bounded convex figures can be only a straight line segment, the boundary of each of the figures $Q_{2m,2n}$ consists of several straight line segments. Thus, all the figures $Q_{2m,2n}$ are polygons. Since the polygons $Q_{2m,2n}$ are central-symmetric, they have an even number of sides (the boundary of such a polygon may be divided into several pairs of symmetric sides). Obviously, the number of sides of $Q_{2m,2n}$ is not less than 4.

Two cases may occur in covering the plane with the polygons $Q_{2m,2n}$:

Case 1. At least one side of the polygon $Q_{0,0}$ adjoins two or more sides of other polygons $Q_{2m,2n}$.

Case 2. Each side of polygon $Q_{0,0}$ adjoins one side of one of the neighboring polygons.

First we shall examine Case 1.

Let us establish a positive direction on the boundaries of polygon $Q_{0,0}$ and all polygons $Q_{2m,2n}$, that is, the direction in which we would travel if we were to make a counterclockwise journey around the polygon. Then a common side of two polygons is assigned opposite directions by the two polygons.

Now let a side a of the polygon $Q_{0,0}$ border at least two other polygons $Q_{2m,2n}$ and $Q_{2m',2n'}$ (Fig. 67). Let B be their common vertex, with AB a side of the first polygon and BC a side of the second polygon.

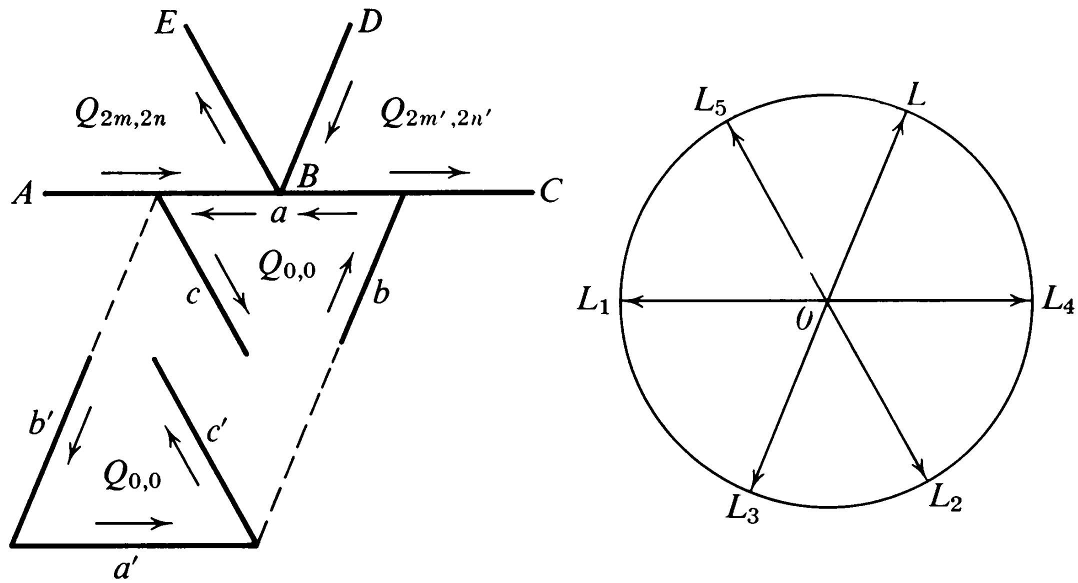


Fig. 67

At vertex B, the side BC of the polygon $Q_{2m',2n'}$ meets a side DB which precedes it in going around this polygon in the positive direction. Similarly, at vertex B the side AB of the polygon $Q_{2m,2n}$ meets a side BE which follows the side AB in going around the boundary of $Q_{2m,2n}$ in the positive direction.

Now let a' be the side of $Q_{0,0}$ symmetric to a with respect to the center O. The translation sending $Q_{2m,2n}$ into $Q_{0,0}$ sends AB into a' and BE into some side c' of polygon $Q_{0,0}$, where c' follows a' in making a circuit of the boundary of $Q_{0,0}$ in the positive direction. Similarly, the translation which carries $Q_{2m',2n'}$ into $Q_{0,0}$ sends DB into a side b' of the polygon $Q_{0,0}$, where b' precedes the side a'. $Q_{0,0}$ also has sides b and c symmetric to b' and c' with respect to the center O.

Now we notice that the side BE cannot lie outside the angle ABD, since it would then be inside the angle CBD and the polygons $Q_{2m,2n}$ and $Q_{2m',2n'}$ would have to intersect.

But BE cannot lie inside angle ABD, as we shall now show. In going around the boundary of the polygon $Q_{0,0}$ in the positive direction, we first pass sides b and a, then side c, and later (immediately or after a series of intermediate sides) side b', then sides a' and c', and so on. Draw the vectors $\overrightarrow{OL}$, $\overrightarrow{OL_1}$, $\overrightarrow{OL_2}$, . . . of length 1 and having the same directions as sides b, a, c, . . . , b', a', c', ... Their end points L, L_1, L_2, . . . , L_3, L_4, L_5, . . . lie on a circle of radius 1. Since the polygon is convex, it follows from the discussion in section 6 that going around this circle in the positive direction, we must pass the vertices L, L_1, L_2, . . . , L_3, L_4, L_5, . . . in order. But this is not possible with sides b, b', c, and c' in the positions shown in Figure 67.

Only one possibility remains—the side BE coincides with the side BD. In this case side b (and side b', symmetric to it) is parallel to side c (and to the corresponding side c') of the same polygon. But a convex polygon cannot have more than two mutually parallel sides. Then b must coincide with c' and c with b', and polygon $Q_{0,0}$ is a parallelogram.

Also notice that since each of the sides AB and BC is equal to the side a, it is not possible for more than two sides of the polygons $Q_{2m,2n}$ to adjoin $Q_{0,0}$ along one side. Thus, in Case 1 the polygons $Q_{2m,2n}$ are arranged as in Figure 68.

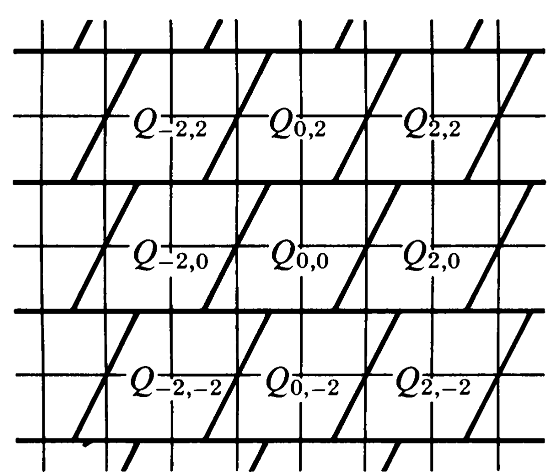


Fig. 68

Now we shall consider Case 2.

The polygon $Q_{0,0}$ adjoins each of its neighboring polygons $Q_{2m,2n}$ along an entire side. Let the side $a = BA$ of the polygon $Q_{0,0}$ coincide with the side $\bar{a} = AB$ of the polygon $Q_{2m,2n}$ (Fig. 69). Let

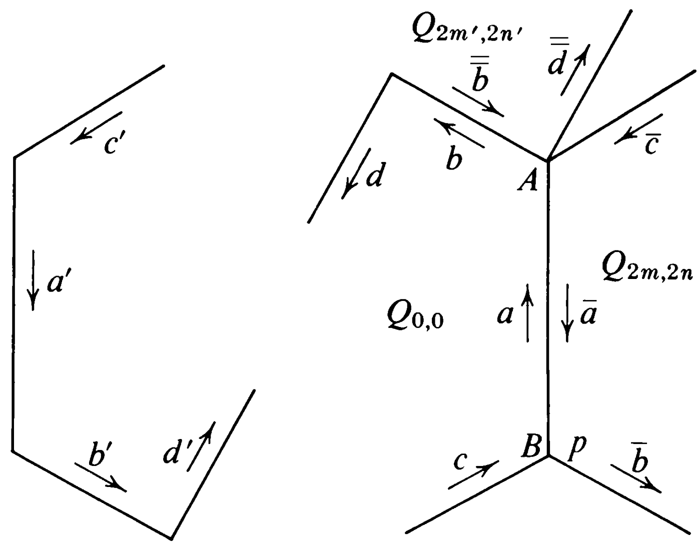


Fig. 69

a' denote the side of $Q_{0,0}$ symmetric to a with respect to the center O; let $\bar{b}$ and $\bar{c}$ denote the sides of polygon $Q_{2m,2n}$ adjacent to $\bar{a}$.

It is possible to translate $Q_{2m,2n}$ to coincide with $Q_{0,0}$ so that side $\bar{a}$ of $Q_{2m,2n}$ coincides with a', and sides $\bar{b}$ and $\bar{c}$ coincide with the sides b' and c' of $Q_{0,0}$, adjoining a'. Let b and c denote the sides of $Q_{0,0}$ adjoining a and symmetric to b' and c' with respect to the center O.

Let us now consider the polygon $Q_{2m',2n'}$ with a side $\bar{\bar{b}}$ coinciding with the side b of the polygon $Q_{0,0}$. The polygon $Q_{2m',2n'}$ has a side $\bar{\bar{d}}$ adjoining the side $\bar{\bar{b}}$ in the vertex A. If we translate $Q_{2m',2n'}$ to coincide with $Q_{0,0}$, then $\bar{\bar{b}}$ coincides with b' and $\bar{\bar{d}}$ with a side d' of polygon $Q_{0,0}$ parallel to it.

We shall denote the side of $Q_{0,0}$ symmetric with d' with respect to O by d. There are two possibilities.

First, it is possible that the side $\bar{\bar{d}}$ lies within the angle made by the sides $\bar{\bar{b}}$ and $\bar{c}$ at the vertex A. Assuming that $\bar{\bar{d}}$ is not an extension of $\bar{a}$ through the vertex A and repeating the reasoning we applied above, we obtain a contradiction of the convexity of the polygon $Q_{0,0}$ (the directions of the sides c, a, b, d, . . . , c', . . . taken in order

are such that $Q_{0,0}$ cannot be a convex polygon). Hence, we must assume that $\bar{\bar{d}}$ is an extension of $\bar{a}$. Then d coincides with a', since a convex polygon cannot have more than two mutually parallel sides. Then b must coincide with c', d' with a, b' with c, and $Q_{0,0}$ is a parallelogram adjoining the neighboring parallelograms along entire sides (Fig. 70).

Second, it is possible that the side $\bar{\bar{d}}$ coincides with the side $\bar{c}$. Then the sides d and d' of the polygon $Q_{0,0}$, parallel to $\bar{\bar{d}}$, must be parallel to the sides c' and c of the same polygon. But a convex polygon cannot have more than two mutually parallel sides, and so d' coincides with c, and the side d, symmetric to it, coincides with c'. Then the polygon $Q_{0,0}$ has six sides—a, b, $d = c'$, a', b', $d' = c$ (Fig. 69). The result is shown in Figure 71.

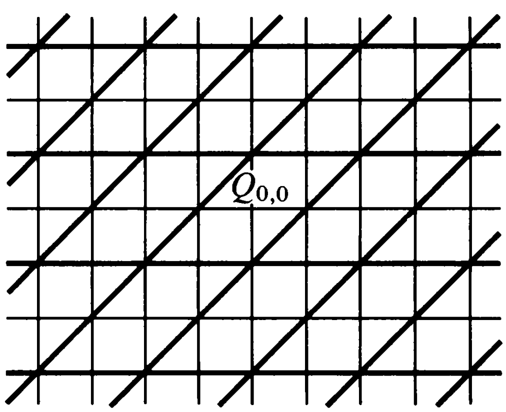


Fig. 70

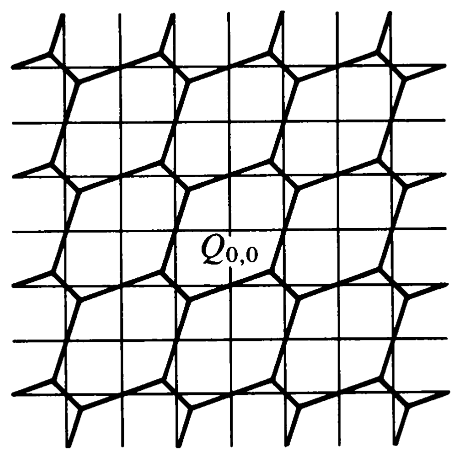


Fig. 71

If the directions of adjacent sides b and c' (b' and c) happen to coincide, then we no longer have Case 2, but Case 1 as shown in Figure 68. Thus, in Case 2 polygon $Q_{0,0}$ is a parallelogram or a central-symmetric hexagon.

Combining Cases 1 and 2, we obtain the following result. The figures $Q_{2m,2n}$, filling the plane without gaps and without overlapping, are either central-symmetric hexagons adjoining each other along entire sides (Fig. 71), parallelograms adjoining each other along entire sides (Fig. 70), or, finally, parallelograms arranged as shown in Figure 68.

Combining these results with the results of section 15, we have:

THEOREM 6. *A central-symmetric plane convex figure of maximal area with center at a lattice point of a lattice of integers and not covering any other lattice points is a parallelogram or a central-symmetric hexagon of area* 4.

The central-symmetric hexagon and parallelogram are called *parallelogons,* and it is possible to cover the entire plane by translating these figures so that they adjoin each other along entire sides.

The three-dimensional analog of a parallelogon is the *parallelohedron;* that is, a convex polyhedron with the property that the whole space may be covered by the translated figures so that they adjoin each other along entire faces. The problem of finding all parallelohedra was posed and solved (in 1885) by the famous Russian crystallographer E. S. Fedorov (1853–1919). They are shown in Figures 72–76. These polyhedra are the central-symmetric figures of maximal volume in an integral lattice.

The parallelohedra are:

(1) Parallelepiped (Fig. 72).

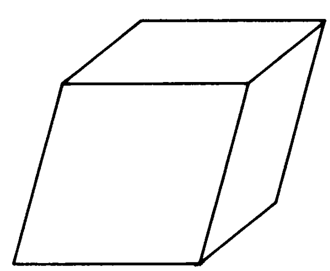


Fig. 72

(2) Prism with a central-symmetric hexagon as base (Fig. 73).

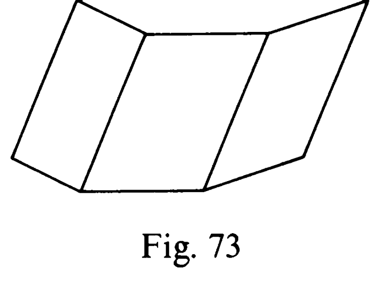


Fig. 73

(3) A dodecahedron (Fig. 74) with faces that are hexagons and rhombuses.

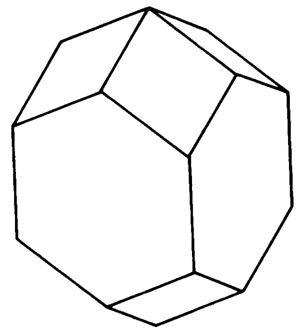


Fig. 74

(4) Rhombic dodecahedron (Fig. 75).

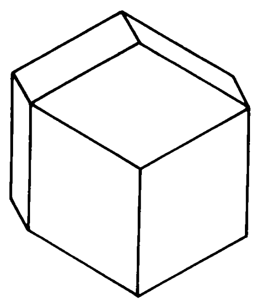


Fig. 75

(5) Truncated octahedron (Fig. 76).

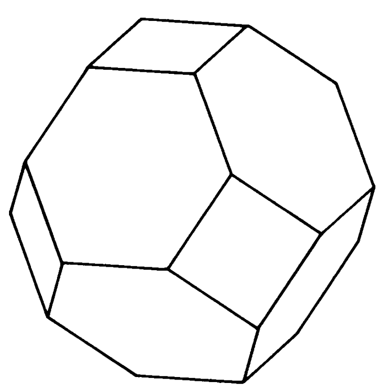


Fig. 76

Note. See also pages 68–74 of *Regular Polytypes,* 2d ed. (New York: Macmillan, 1963).

3. Networks and Convex Polyhedra

17. VERTICES (NODES), FACES (REGIONS), AND EDGES (LINES); EULER'S THEOREM

The theory of polyhedra, in particular of convex polyhedra, is one of the most fascinating chapters of geometry. It was studied in ancient times, and Book XIII of Euclid's *Elements* is devoted to the five regular convex polyhedra. Archimedes, in his work *On Polyhedra,* added the so-called semiregular polyhedra (cf. section 39). At various stages in the development of mathematics, geometers have returned to the theory of convex polyhedra and have discovered new fundamental facts about it.

We begin by stating a remarkable theorem of Euler (1706–1783).

EULER'S THEOREM. *For any convex polyhedron the number of vertices plus the number of faces minus the number of edges is equal to* 2; *that is,*

$$m + n - l = 2,$$

where l denotes the number of edges of the polyhedron, m the number of vertices, and n the number of faces.[1]

In the following table we verify Euler's equation for the regular convex polyhedra and for the five parallelohedra defined in section 16.

Figure	l	m	n	$m + n - l$
Regular tetrahedron (Fig. 77)	6	4	4	$4 + 4 - 6 = 2$
Cube (parallelepiped) (Fig. 78, 72)	12	8	6	$8 + 6 - 12 = 2$
Regular octahedron (Fig. 79)	12	6	8	$6 + 8 - 12 = 2$
Regular dodecahedron (Fig. 80)	30	20	12	$20 + 12 - 30 = 2$
Regular icosahedron (Fig. 81)	30	12	20	$12 + 20 - 30 = 2$
Hexagonal prism (Fig. 73)	18	12	8	$12 + 8 - 18 = 2$
A dodecahedron (Fig. 74)	28	18	12	$18 + 12 - 28 = 2$
Rhombic dodecahedron (Fig. 75)	24	14	12	$14 + 12 - 24 = 2$
Truncated octahedron (Fig. 76)	36	24	14	$24 + 14 - 36 = 2$

[1] For generalization, see section 39.

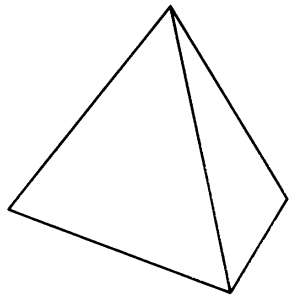


Fig. 77

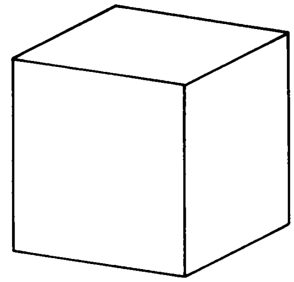


Fig. 78

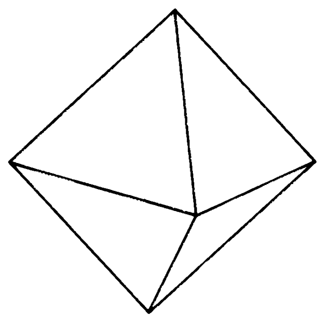


Fig. 79

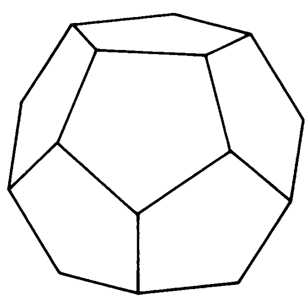


Fig. 80

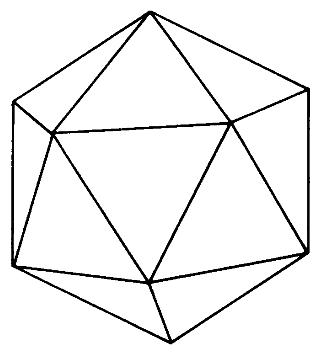


Fig. 81

[Note that in this chapter we shall use the word "line" for either a straight or a curved line.]

DEFINITION. *A* network on a surface *consists of a finite set of points (called* nodes), *lines, and regions of the surface satisfying the following conditions:*

(1) *There are nodes at the ends of the lines.*
(2) *Nodes exist only at the ends of lines.*
(3) *Each line contains the nodes at its ends.*
(4) *The only points which may be common to two lines are nodes.*
(5) *Lines do not intersect themselves.*
(6) *The regions are exactly the parts into which the lines divide the surface.*

EXAMPLES:

1. In Figure 82, the points $A, A_1, A_2, \ldots, A_8, A_9$ are the nodes of a network. The lines $a_1, a_2, \ldots, a_{14}$ are its lines, and the regions $Q_1, Q_2, \ldots, Q_6$ are its regions.

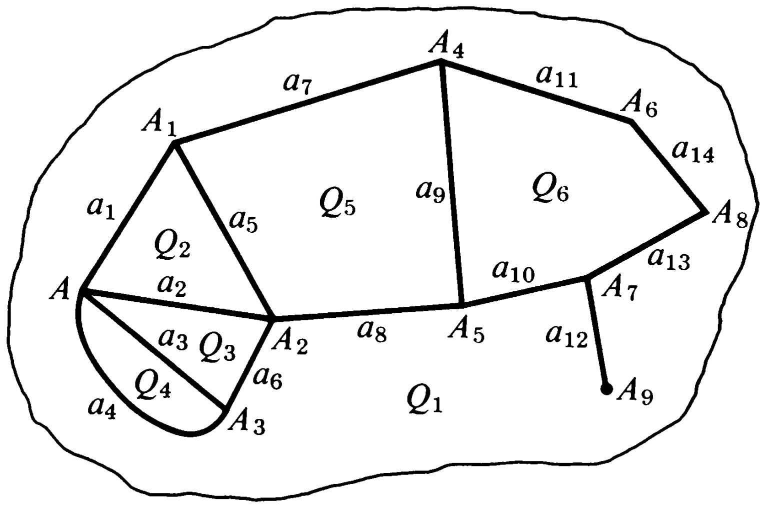


Fig. 82

2. We have a natural example of a network on the surface of any polyhedron. Its vertices are the nodes, its edges are the lines, and its faces are the regions of the network. We shall be especially interested in this network.

3. The simplest possible network, called a *prime* network, consists of a single point A on the surface. It has no lines. There is only one region, the entire surface S.

4. A closed polygon on a surface S is an example of a network consisting of k nodes $A_1, A_2, \ldots, A_k$ (the vertices of the polygon) and k lines $A_1A_2, A_2A_3, \ldots, A_{k-1}A_k, A_kA_1$ (the sides of the polygon). Two sides cannot have common points other than their end points. The closed polygon divides the surface S into two parts, the regions of our network.

DEFINITION. *A network is a* connected network *if it is possible to go from one of its nodes to any other node by moving along its lines.*

All networks in the above examples are connected. If k is the least number of lines of the network needed to provide a path from node A to node B, then we shall say that B is k *steps* from A. For example, in Figure 82 node A_9 is 4 steps from node A.

Now we shall generalize Euler's theorem to networks on convex surfaces. Let m denote the number of nodes of a network, n, the number of regions, and l, the number of lines. We have already stated that for a network made up of the vertices, faces, and edges of a regular polyhedron or a parallelohedron, the numbers m, n, and l are related by the equation

$$m + n - l = 2. \tag{1}$$

It is easy to convince oneself that equation (1) holds for all the examples of connected networks we have considered. In particular, for a prime network (Example 3),

$$m = 1, \qquad n = 1, \qquad l = 0, \qquad m + n - 1 = 2. \tag{2}$$

For the network in Example 4,

$$m = l = k, \qquad n = 2, \qquad m + n - l = 2.$$

The expression $m + n - l$ is called the *Euler characteristic of the network.*

THEOREM 1. *The Euler characteristic of any connected network on a convex surface is equal to* 2:

$$m + n - l = 2.$$

A special case of this theorem is the theorem of Euler, stated earlier in this section, on the relationship between the numbers of vertices, faces, and edges of a polyhedron (since they are the nodes, regions, and lines of the corresponding network on the surface of the polyhedron).

18. PROOF OF THE THEOREM FOR CONNECTED NETWORKS

We shall now prove Theorem 1 of section 17, of which Euler's theorem is a special case.

We need two operations by which to transform connected networks into more complex connected networks. These operations will be called *supplementations* because they involve adding supplementary lines to the given networks.

(1) Supplement of the first kind: From a node A of a network K we draw a new line AB (having no point other than its end point A in common with any other line of the network) so that its other end B does not belong to the network. (For example, add line AB to the

network of Figure 83 as shown by the dotted line.) Thus our network is transformed into a new connected network K_1.

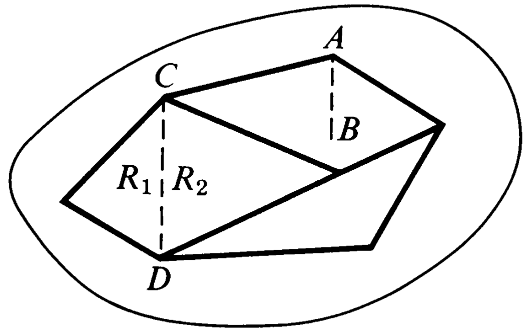


Fig. 83

In comparison with K, network K_1 has one more line, AB, and one more node, B. The number of regions is unchanged by the addition of line AB. If for network K the number of nodes, regions, and lines are equal to m, n, and l, then for network K_1 they are equal to $(m + 1)$, n, and $(l + 1)$ respectively. *The Euler characteristics of both networks are equal:*

$$m + n - l = (m + 1) + n - (l + 1).$$

(2) Supplement of the second kind: Connect two nodes C and D of a network K by a line CD not having points other than its end points in common with any other lines of the network. (Such a line is shown by a dotted line in Figure 83.) We obtain a new network K'. Line CD lies entirely in one region R of network K and divides it into two regions, R_1 and R_2, of network K'.

In comparison with K, network K' has one more line and one more region. (In place of one region R, there are two, R_1 and R_2.) Both networks have the same number of nodes. If m, n, and l denote the numbers of nodes, regions, and lines of network K, then for network K' the corresponding numbers are m, $(n + 1)$, and $(l + 1)$. *The Euler characteristics of both networks are equal:*

$$m + n - l = m + (n + 1) - (l + 1).$$

Thus, we arrive at the lemma:

LEMMA 1. *Supplements of the first and second kind do not change the Euler characteristic of a network on a convex surface.*

We can also prove the following:

LEMMA 2. *Any connected network on a convex surface can be obtained from a prime network by adding supplements of the first and second kind in order.*

Proof. Let K be a connected network on a convex surface P and let A be one of its nodes (Fig. 82). Let K_0 denote the prime network consisting of point A. Consider all lines of K coming out of node A (for example, lines a_1, a_2, a_3, a_4 in Figure 82). Begin with node A alone. Then annex lines a_1, a_2, a_3, and a_4, in order, as supplements of the first and second kinds. (For example, in Figure 82 we first adjoin lines a_1, a_2, and a_3 as supplements of the first kind, then a_4 as a supplement of the second kind.) We obtain a new network K_1. (In Figure 82, it contains lines a_1, a_2, a_3, a_4 and nodes A, A_1, A_2, A_3). The nodes of K_1 different from A are *one* step away from A.

Now we add a new sequence of supplements of the first and second kinds, adding to K_1 the lines of K which have one or two end points among the nodes of K_1, but which do not themselves belong to K_1 (lines a_5, a_6, a_7, and a_8 in Figure 82). We obtain a new network K_2. The nodes of K_2 which do not belong to K_1 (nodes A_4, A_5 in Figure 82) are the nodes of the original network K, which are two steps away from A.

Next, we make a third sequence of supplements of the first and second kinds, adding to K_2 those lines of K which do not belong to K_2 but have one or two end points belonging to K_2 (lines a_9, a_{10}, a_{11} in Figure 82). We obtain network K_3. Its nodes not belonging to K_2 (nodes A_6, A_7 in Figure 82) are the nodes of K three steps away from A.

We continue in this manner to obtain networks $K_4, K_5, \ldots$, which contain all nodes of K which are 4, 5, ... steps away from A, and so on. All nodes of network K are within n steps of A (because there are only n lines in K). So, by repeating the above process at most $n + 1$ times, we obtain a network containing all the nodes and all the lines of K. Thus we obtain the entire network K, and the lemma is proved.

Proof of Theorem 1. The Euler characteristic of a prime network K_0 is equal to 2 (equation (2), section 17). The network K is obtained from K_0 by a sequence of supplements of the first and second kinds (Lemma 2), each of which leaves the Euler characteristic unchanged (Lemma 1). Therefore, the Euler characteristic of network K is also equal to 2, and the theorem is proved.

19. DISCONNECTED NETWORKS; INEQUALITIES

DEFINITION. *A* disconnected network *on a convex surface is one that consists of some number, s, of connected networks.*

For example, a network composed of two closed polygons consists of two connected networks and $s = 2$.

THEOREM 2. *The Euler characteristic of a network N on Q is equal to*

$$m + n - l = s + 1, \tag{1}$$

where s is the number of connected networks comprising N. (For $s = 1$ this formula reduces to formula (1) of section 17.)

Proof. Let network K consist of s connected networks K_1, K_2, $\ldots$, K_s. Choose one node from each, $A_1, A_2, \ldots, A_s$. These s points form a network K_0 on the surface Q for which $m = s$ (s nodes), $n = 1$ (one region, the entire surface Q), and $l = 0$. (K_0 has no lines.) The Euler characteristic of network K_0 is equal to

$$m + n - l = s + 1 - 0 = s + 1. \tag{2}$$

Just as for a connected network, we can prove that network K is obtained from K_0 by a sequence of supplements of the first and second kinds, which do not change the Euler characteristic. Then the Euler characteristics of K and K_0 are equal, and by (2), both are equal to $s + 1$.

We shall now consider networks in which at least two lines come from each node and each region is bounded by at least two lines (Fig. 84). (For example, the natural network on the surface of a regular polyhedron, composed of its vertices, faces, and edges.) The lines of such a network may be divided into two types: lines of the first kind, which border two regions (such as an edge of a convex polyhedron in its natural network), and lines of the second kind, which border only one region, that is, having the same region

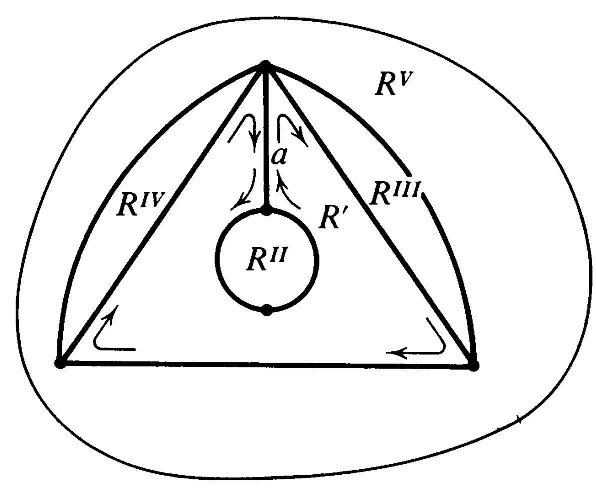


Fig. 84

on both sides, such as line a and region R' in Figure 84. In making the circuit of such a region in either direction (for example, if we go around the boundary of R' in the clockwise direction), we travel along a line a of the second kind twice. We shall consider such a line to be composed of a pair of coincident lines.

We shall call a region of a network a k-gon if its boundary consists of k lines—"sides"—of the network. A line of the second kind is to be counted twice. For example, region R' in Figure 84 is a 7-gon.

Let n_2 denote the number of 2-gons in the network, n_3 the number of 3-gons, and in general n_k the number of k-gons in the network. If, as before, m, l, and n are the numbers of nodes, lines, and regions of the network, then

$$n = n_2 + n_3 + n_4 + \cdots + n_k. \tag{3}$$

On the other hand, we shall show that

$$2l = 2n_2 + 3n_3 + 4n_4 + \cdots + kn_k. \tag{4}$$

In fact, the total number of sides of 2-gons in the network is $2n_2$, of 3-gons is $3n_3$, and so on. Consequently, the right side of equation (4) is the total number of sides of all regions of the network. But each line of the network was counted twice. A line of the first kind lies on the boundary of two regions of the network and is counted as a side of both regions. A line of the second kind is considered to be a pair of sides of one region. Hence, our sum is equal to twice the number of lines in the network. Therefore (4) follows.

By equation (1) of this section, since $s \geq 1$, we have:

$$m + n - l \geq 2$$

or

$$m - 2 \geq l - n. \tag{5}$$

Multiplying inequality (5) termwise by 4, we obtain the formula

$$4m - 8 \geq 4l - 4n = 2 \cdot 2l - 4n. \tag{6}$$

From formulas (3), (4), and (6) it follows that

$$4m - 8 \geq 2(2n_2 + 3n_3 + 4n_4 + \cdots + kn_k) - 4(n_2 + n_3 + n_4 + \cdots + n_k),$$

or

$$4m - 8 \geq 2n_3 + 4n_4 + 6n_5 + 8n_6 + \cdots + (2k - 4)n_k. \tag{7}$$

20. CONGRUENT AND SYMMETRIC POLYHEDRA; CAUCHY'S THEOREM

It is possible to transform a convex polygon into another one by changing its angles without changing the lengths of its sides. For example, square $ABCD$ in Figure 85 may be transformed into

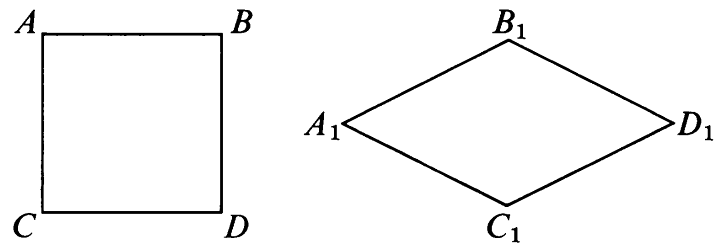


Fig. 85

rhombus $A_1B_1C_1D_1$ so that the sides of the square and the rhombus have equal length. The right angles of the square are transformed into two acute and two obtuse angles of the rhombus.

It is natural to pose the analogous question for polyhedra. Is it possible to change the dihedral angles of a polyhedron without changing its faces? The famous French mathematician Augustin Cauchy (1789–1857) has given the negative answer to this question. In his article published in 1813 in the *Journal de l'École Polytechnique,* he proved the following theorem.

CAUCHY'S THEOREM. *Two convex polyhedra with corresponding faces equal (congruent) and equally arranged have equal dihedral angles between corresponding faces.*

Cauchy's theorem may be reworded as follows: Two convex polyhedra with corresponding faces equal and equally arranged are either congruent or symmetric.

First, we remark that this theorem ceases to be true if we drop the restriction to convex polyhedra. Consider, for example, hexahedra Q and Q_1 in Figure 86. Convex hexahedron Q consists of two tetrahedra $ABCD$ and $ABCE$ bordering each other in their common face ABC. Nonconvex hexahedron Q_1 is obtained from tetrahedron $A_1B_1C_1D_1$, congruent to $ABCD$, by "carving out" tetrahedron $A_1B_1C_1E_1$, symmetric to $ABCE$. All six pairs of corresponding faces of the two hexahedra are congruent, but, obviously, the dihedral angles along edges AB and A_1B_1 (also along BC and B_1C_1, CA and C_1A_1) are not equal.

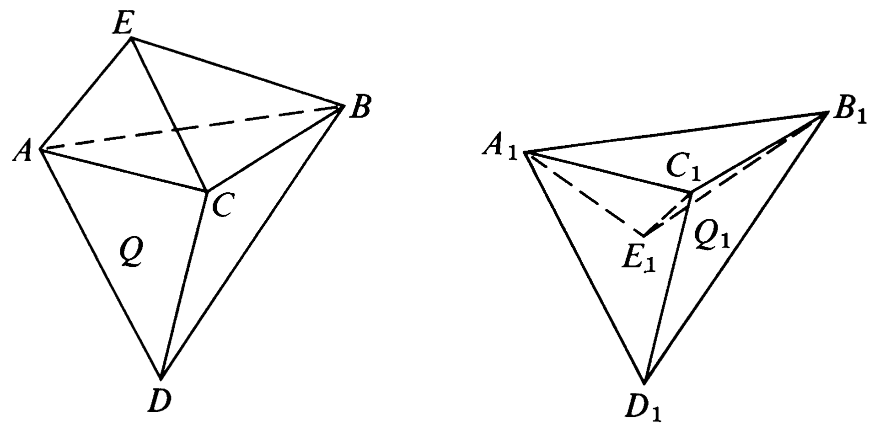


Fig. 86

We shall give Cauchy's own proof of this theorem. It is based on a series of lemmas treating transformations of plane and spherical convex polygons. Since the proofs for both cases are entirely the same, we shall consider both cases simultaneously.

First we point out that Cauchy's proof of Lemma 1 given below contains an inaccuracy that was noticed and corrected by Steinitz (see section 22).

LEMMA 1. *Let us transform a convex polygon (plane or spherical) $A_1A_2 \ldots A_n$ into another convex polygon $A_1'A_2' \ldots A_{n-1}'A_n'$ so that the lengths of the sides $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$ are unchanged. If under this transformation the angles at vertices $A_2, A_3, \ldots, A_{n-1}$ either all increase or some of them increase and the remainder are unchanged, then the length of side A_nA_1 increases. On the contrary, if the angles at vertices $A_2, A_3, \ldots, A_{n-1}$ all decrease or if some of them decrease and the remainder are unchanged, then the length of side A_nA_1 decreases.*

Proof. The lemma is obvious for triangles. If in triangle ABC side AB and BC are unchanged and the angle at B increases, then the length of side AC increases. (This follows from a well-known theorem about plane and spherical triangles: If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second.) Similarly, decreasing the angle at B without changing the lengths of sides AB and BC causes the length of side AC to decrease.

Now we consider polygon $A_1A_2 \ldots A_{n-1}A_n$ (Fig. 87).

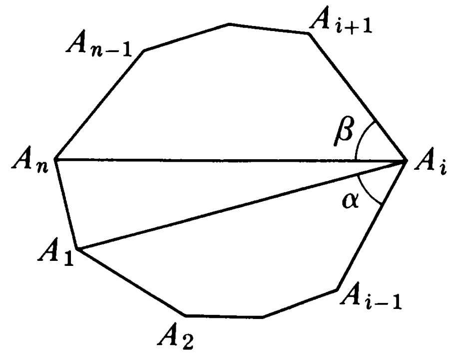


Fig. 87

Assume that the lengths of all its sides except A_nA_1 remain constant, and only one of the angles at vertices $A_2, A_3, \ldots, A_{n-1}$ changes. In particular, let angle $A_{i-1}A_iA_{i+1}$ at vertex A_i increase. Connect A_i with vertices A_1 and A_n by the segments A_1A_i and A_iA_n. Since in polygons $A_1A_2 \ldots A_{i-1}A_i$ and $A_iA_{i+1} \ldots A_{n-1}A_n$, the length of each side and the angles between them do not change, then each polygon is carried into a congruent polygon. In particular, the lengths of segments A_1A_i and A_iA_n, and the measures of angles $\alpha = \angle A_1A_iA_{i-1}$ and $\beta = \angle A_{i+1}A_iA_n$ do not change. But $\angle A_1A_iA_n = \angle A_{i-1}A_iA_{i+1} - \alpha - \beta$, and so, since α and β do not change, this angle increases with $\angle A_{i-1}A_iA_{i+1}$.

In triangle $A_1A_iA_n$, the lengths of sides A_1A_i and A_iA_n do not change, but the angle at vertex A_i increases. Therefore, the length of side A_1A_n also increases. Now suppose that more than one of the angles at vertices $A_2, \ldots, A_{n-1}$ of our polygon increase (for example, at A_i, A_k, ...), and the others remain constant. At first we increase the angle at A_i, leaving the remaining angles unchanged, causing side A_1A_n to increase. Then, not changing any other angles, we increase the angle at A_k. Side A_1A_n again increases, and so forth. As a result, if all the angles at vertices $A_2, \ldots, A_{n-1}$ are increased, or part are increased and the rest left unchanged, side A_1A_n increases. On the other hand, if all or part of these angles are decreased, then side A_1A_n decreases. This completes Cauchy's proof of Lemma 1.

COROLLARY. *If a convex plane or spherical polygon $A_1A_2 \ldots A_n$ is transformed into another without changing the lengths of its sides, but one of its angles is increased, then at least one of its remaining angles must decrease.*

Proof. Suppose that the angle at A_2 increases. If each of the remaining angles either increases or remains constant, then by Lemma 1 side A_1A_n must increase. But since it remains unchanged, one of the remaining angles at vertices A_3, A_4, . . . , A_{n-1} must decrease.

Remark. Let us mark some of the vertices of a convex polygon with plus and minus signs (Fig. 88). In making the circuit of the

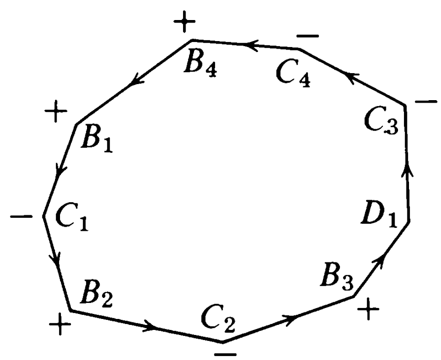


Fig. 88

vertices in order, we shall proceed several times from a vertex with a plus sign to one with a minus sign (for example, from B_1 to C_1) and conversely. In doing so we say that the sign changes from plus to minus or from minus to plus, respectively. Obviously, in making a circuit of the vertices in order, we go from plus to minus just as many times as from minus to plus. Therefore the total number of changes of sign is even (it is six for Fig. 88).

LEMMA 2. *Let us transform a convex plane or spherical polygon into another without changing the lengths of its sides. Mark the vertices whose angles increase with plus signs, and the vertices whose angles decrease with minus signs (vertices whose angles do not change are left unmarked). Then, in making a circuit of the vertices in order, we shall find at least two changes of sign from plus to minus and at least two changes from minus to plus (hence, at least* 4 *changes in all).*

Proof. First, if there is even one vertex marked plus, then it follows that there is at least one vertex marked minus (Corollary to Lemma 1). Now suppose that there is only one change of sign from plus to minus and hence one from minus to plus (see the preceding Remark). Then the vertices may be divided into two groups such that each group contains, in order, all the vertices having the same

sign, and possibly some vertices without signs; for example, groups $C_1C_2D_1C_3$ and $B_1B_2B_3$ in Figure 89.

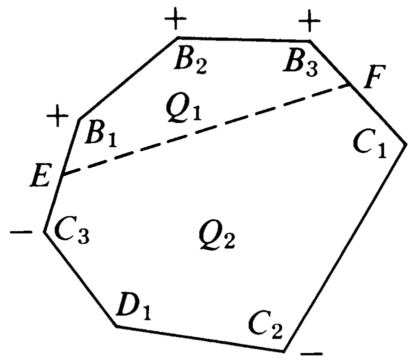


Fig. 89

Choose an arbitrary point E on the polygon lying between two vertices with different signs, for example between B_1 and C_3, and a point F lying on the other side of the polygon also between vertices with different signs, for example between B_3 and C_1. Segment EF divides our convex polygon into two convex polygons: polygon Q_1, which carries all the vertices marked plus, and Q_2 carrying all the vertices marked minus. Applying Lemma 1 to polygon Q_1, we must conclude that the length of side EF must increase. Applying the same lemma to Q_2 we find that the length of side EF must decrease. This contradiction completes the proof of Lemma 2.

COROLLARY. *If a convex polygon is transformed into another without changing the lengths of the sides, then either all the angles remain unchanged or at least* 4 *angles are changed: at least two increase and at least two decrease.*

LEMMA 3. *Let a convex solid angle T be transformed into another convex solid angle without changing its face angles at the vertex O (only the dihedral angles may change). Mark by a plus sign the edges through O along which the dihedral angle increases, and by a minus sign the edges along which the dihedral angle decreases. If even one dihedral angle is changed under our transformation, then in going around vertex O we shall find at least two changes of sign from plus to minus and at least two changes from minus to plus.*

Proof. Construct a sphere S with center at O (Fig. 90). The in-

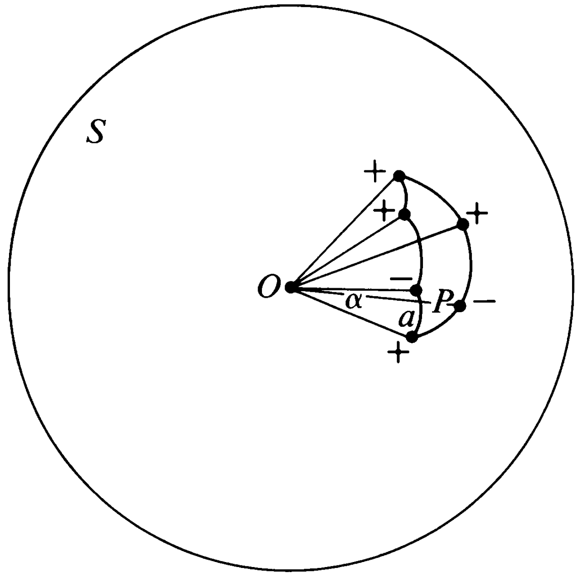


Fig. 90

tersection of the surface of the sphere S with the solid angle T is some spherical convex polygon P. Each face K of angle T gives a side a of the spherical polygon. The length of a is determined by the angle α in face K at vertex O. The dihedral angle between adjacent faces K and K_1 determines the angle between the corresponding sides a and a_1 of polygon P. Hence, any change in the solid angle corresponds to a change in the polygon. If the changes in the solid angle satisfy the conditions of the lemma, then the sides of the polygon do not change, and its angles increase or decrease together with the corresponding dihedral angles of T. Since Lemma 2 is valid for convex spherical polygons, Lemma 3 must be true for convex solid angles.

21. PROOF OF CAUCHY'S THEOREM

Now we proceed to prove Cauchy's theorem.

Proof. Consider two polyhedra Q and Q_1 with congruent and similarly arranged corresponding faces. Mark with a plus sign each edge of Q along which the dihedral angle is greater than the corresponding dihedral angle of Q_1, and mark with a minus sign each edge of Q along which the dihedral angle is smaller than the corresponding dihedral angle of Q_1. Do not mark the remaining edges (along which the dihedral angle of Q is equal to the corresponding dihedral

angle of Q_1). If there are no edges marked with signs $+$ or $-$, then all corresponding dihedral angles of Q and Q_1 are equal. In this case the theorem is true.

We shall assume that there are edges marked by $+$ and $-$ signs and show that this assumption leads to a contradiction of formula (7) of section 19.

Suppose that several edges marked with $+$ or $-$ signs intersect at some vertex of Q. By Lemma 3 of section 20, there are at least four edges marked with $+$ or $-$ signs through this vertex. These edges are arranged so that in going around the vertex once, we find a total of at least 4 changes of signs from $+$ to $-$ or from $-$ to $+$.

The edges marked with $+$ or $-$ and the vertices they terminate in, form a network K on the surface of Q. Each region in this network consists of one or more faces of Q. Going around each node of K, we find some even number of changes of sign, (greater than or equal to 4), as was explained above. Let M denote the sum of the numbers of changes of sign obtained by going around all the nodes of network K. Then

$$M \geq 4m \tag{1}$$

where m is the number of nodes of K.

On the other hand, in going around the boundary of any region in the network we always find an even number of changes of sign. Consequently, in going around a k-sided region this number is not greater than k if k is even, and not greater than $k-1$ if k is odd. In going around a 3-sided region the number of changes of sign is not greater than 2; in going around a 4-sided region it is not greater than 4; a 5-sided region, also 4; 6-sided and 7-sided, not greater than 6; and so on. Let the numbers of 3-sided, 4-sided, ... regions of K be given by $a_3, a_4, \ldots$. Since we obtain the same number M of changes of sign by adding the changes in going around each region, we know:

$$M \leq 2a_3 + 4a_4 + 4a_5 + 6a_6 + 6a_7 + 8a_8 + \cdots. \tag{2}$$

From this and formula (1) we obtain:

$$4m \leq 2a_3 + 4a_4 + 4a_5 + 6a_6 + 6a_7 + 8a_8 + \cdots. \tag{3}$$

On the other hand, in section 19 we proved (7), which may be rewritten as:

$$4m - 8 \geq 2a_3 + 4a_4 + 6a_5 + 8a_6 + 10a_7 + \cdots. \tag{4}$$

Subtracting formula (3) from formula (4) termwise, we obtain:

$$-8 \geq 2a_5 + 2a_6 + 4a_7 + 4a_8 + 6a_9 + \cdots .$$

But since the right-hand side is a nonnegative number, then $-8 \geq 0$. We have reached a contradiction, and the theorem is proved.

COROLLARY. *The only possible continuous transformation of a convex polyhedron under which all faces remain congruent to themselves is a motion of the polyhedron as a rigid solid.*

In fact, by Cauchy's theorem, the convex polyhedron must either remain congruent to itself or become a symmetric image of itself. Since the transformation was assumed to be continuous, the polyhedron cannot be carried into a symmetric image of itself (unless, of course, it is symmetric to itself). Hence our polyhedron undergoes a continuous motion, remaining at all times congruent to its original self, that is, moves as a rigid solid.

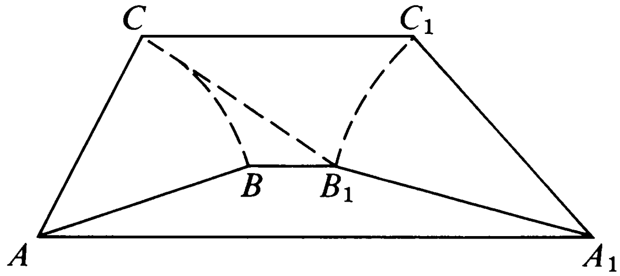
22. STEINITZ' CORRECTION OF CAUCHY'S PROOF

Without doubt, Cauchy's original proof is a masterpiece of geometry, but it contains a defect, which Steinitz, a German mathematician, noted and corrected. This defect is in the proof of Lemma 1 of section 20, on which the remainder of Cauchy's proof is based. First (cf. p. 68) it was shown that in a convex n-gon if $n - 1$ sides and all but one of the angles they form remain unchanged, while that angle increases (transforming the convex n-gon into another convex n-gon), then the nth side increases. On this was based the proof of Lemma 1: If all the angles formed by $n - 1$ sides whose lengths do not change increase (or part of them increase and the remainder are unchanged), then the length of the nth side increases. For the proof, we first increased one of the angles while holding the rest constant, then increased another one, and so on. *If this process yields a convex figure,* then, by the above reasoning, the nth side increases each time. Thus Lemma 1 was proved in section 20.

A difficulty arises, however. By increasing one angle of a convex n-gon and holding $n - 1$ sides constant, we may produce a nonconvex polygon, as simple examples will show. But *for nonconvex polygons, increasing some of its angles while holding $n - 1$ sides constant does not necessarily lead to an increase in the nth side.*

Figure 91 shows two trapezoids, ABB_1A_1 and ACC_1A_1 with

Fig. 91



common side AA_1 and pairwise equal corresponding sides $AB = AC$, $A_1B_1 = A_1C_1$. In order to obtain trapezoid ACC_1A_1 from trapezoid ABB_1A_1, we need to increase $\angle BAA_1$ to the size of $\angle CAA_1$ and $\angle B_1A_1A$ to the size of $\angle C_1A_1A$ without changing the lengths of sides AA_1, AB, and A_1B_1. But if we first increase angle BAA_1 to the size of angle CAA_1, preserving angle B_1A_1A and the lengths of sides B_1A_1, AA_1, and AB, then trapezoid ABB_1A is transformed into the *nonconvex* quadrilateral CAA_1B_1, and only after increasing the second angle, B_1A_1A, is it converted back into a convex figure, the trapezoid CAA_1C_1.

Lemma 1 of section 20 is still true, however, and we may continue to use it as the basis of the proof of Cauchy's theorem. Steinitz has replaced the inaccurate proof given previously with a rigorous but longer one, which we shall now give.[1]

Proof of Lemma 1. The proof is by induction. The theorem is known to hold for triangles. We shall assume it is true for convex polygons with $n - 1$ sides and prove that it holds for polygons with n sides ($n \geq 4$).

Suppose that we are given two convex polygons $A_1A_2 \ldots A_n$ and $B_1B_2 \ldots B_n$, where

$$A_1A_2 = B_1B_2, \quad A_2A_3 = B_2B_3, \quad \ldots, \quad A_{n-1}A_n = B_{n-1}B_n, \quad (1)$$

$$\angle A_1A_2A_3 \leq \angle B_1B_2B_3, \quad \angle A_2A_3A_4 \leq \angle B_2B_3B_4, \quad \ldots, \quad \angle A_{n-2}A_{n-1}A_n \leq \angle B_{n-2}B_{n-1}B_n, \quad (2)$$

of which at least one pair of angles $\angle A_{i-1}A_iA_{i+1}$, $\angle B_{i-1}B_iB_{i+1}$ satisfies the strict inequality

$$\angle A_{i-1}A_iA_{i+1} < \angle B_{i-1}B_iB_{i+1}.$$

We must prove the inequality

$$A_nA_1 < B_nB_1.$$

[1] E. Steinitz and H. Rademacher, *Vorlesungen über die Theorie der Polyeder* (Berlin: Julius Springer, 1937).

Case 1. If equality holds in even one of the inequalities (2), then $A_nA_1 < B_nB_1$ is easily proved as follows: let (Fig. 92)

$$\angle A_{j-1}A_jA_{j+1} = \angle B_{j-1}B_jB_{j+1}.$$

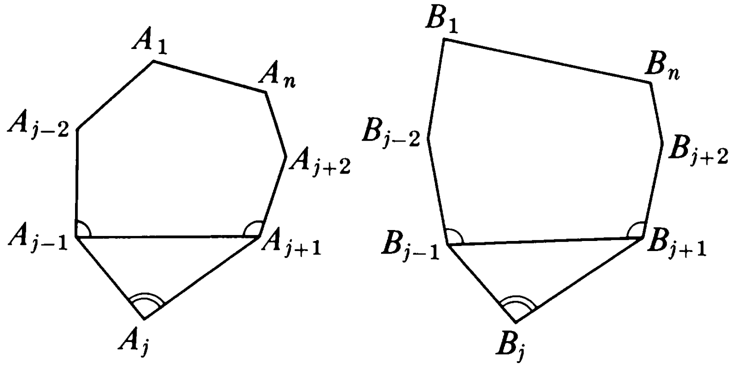


Fig. 92

Then because $A_{j-1}A_j = B_{j-1}B_j$ and $A_jA_{j+1} = B_jB_{j+1}$, triangles $A_{j-1}A_jA_{j+1}$ and $B_{j-1}B_jB_{j+1}$ are congruent. Hence, their third sides are equal,

$$A_{j-1}A_{j+1} = B_{j-1}B_{j+1}, \tag{3}$$

as are the angles

$$\left.\begin{aligned} \angle A_{j+1}A_{j-1}A_j &= \angle B_{j+1}B_{j-1}B_j, \\ \angle A_{j-1}A_{j+1}A_j &= \angle B_{j-1}B_{j+1}B_j. \end{aligned}\right\} \tag{4}$$

Consider polygons $A_1A_2 \ldots A_{j-1}A_{j+1}A_{j+2} \ldots A_n$ and $B_1B_2 \ldots B_{j-1}B_{j+1}B_{j+2} \ldots B_n$. They are convex and have $n-1$ sides. Thus, $n-2$ pairs of corresponding sides are equal (cf. formulas (1) and (3)):

$$A_1A_2 = B_1B_2, \quad \ldots, \quad A_{j-1}A_{j+1} = B_{j-1}B_{j+1}, \quad \ldots,$$
$$A_{n-1}A_n = B_{n-1}B_n. \tag{5}$$

The angles formed by these sides are related by the inequalities

$$\left.\begin{aligned} &\angle A_1A_2A_3 \leq \angle B_1B_2B_3, \quad \ldots, \\ &\qquad\qquad \angle A_{j-2}A_{j-1}A_{j+1} \leq \angle B_{j-2}B_{j-1}B_{j+1}, \\ &\angle A_{j-1}A_{j+1}A_{j+2} \leq \angle B_{j-1}B_{j+1}B_{j+2}, \quad \ldots, \\ &\qquad\qquad \angle A_{n-2}A_{n-1}A_n \leq \angle B_{n-2}B_{n-1}B_n. \end{aligned}\right\} \tag{6}$$

In fact (Fig. 92) we have,

$$\angle A_{j-2}A_{j-1}A_{j+1} = \angle A_{j-2}A_{j-1}A_j - \angle A_{j+1}A_{j-1}A_j,$$
$$\angle B_{j-2}B_{j-1}B_{j+1} = \angle B_{j-2}B_{j-1}B_j - \angle B_{j+1}B_{j-1}B_j.$$

So by formulas (2) and (4),

$$\angle A_{j-2}A_{j-1}A_{j+1} \leq \angle B_{j-2}B_{j-1}B_{j+1};$$

analogously,

$$\angle A_{j-1}A_{j+1}A_{j+2} \leq \angle B_{j-1}B_{j+1}B_{j+2}.$$

The remaing inequalities of (6) are found among the inequalities (2).

Assuming that the theorem holds for polygons with $n - 1$ sides, we conclude from formulas (5) and (6) that $A_nA_1 < B_nB_1$. That is what we had to prove.

Case 2. If all the inequalities of (2) are strict inequalities:

$$\angle A_1A_2A_3 < \angle B_1B_2B_3, \quad \ldots, \quad \angle A_{n-2}A_{n-1}A_n < \angle B_{n-2}B_{n-1}B_n.$$

Let γ be an angle whose value is between $\angle A_{n-2}A_{n-1}A_n$ and $\angle B_{n-2}B_{n-1}B_n$. Consider the n-gons $A_1A_2 \ldots A_n'$ and $B_1B_2 \ldots B_n'$ (Fig. 93) where $\angle A_{n-2}A_{n-1}A_n' = \angle B_{n-2}B_{n-1}B_n' = \gamma$, $A_{n-1}A_n = A_{n-1}A_n'$, and $B_{n-1}B_n = B_{n-1}B_n'$.

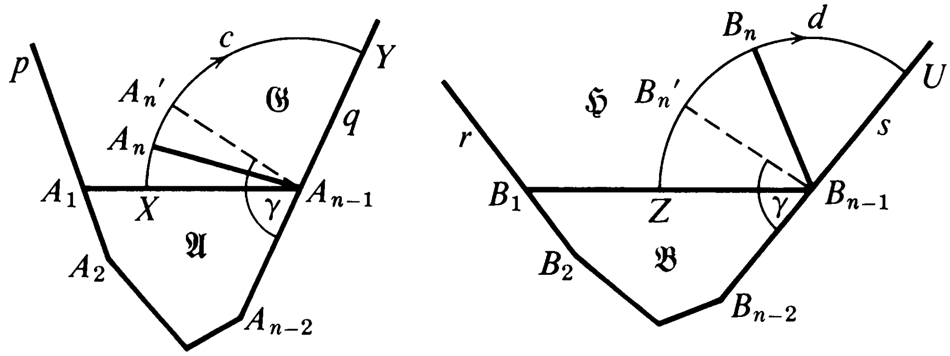
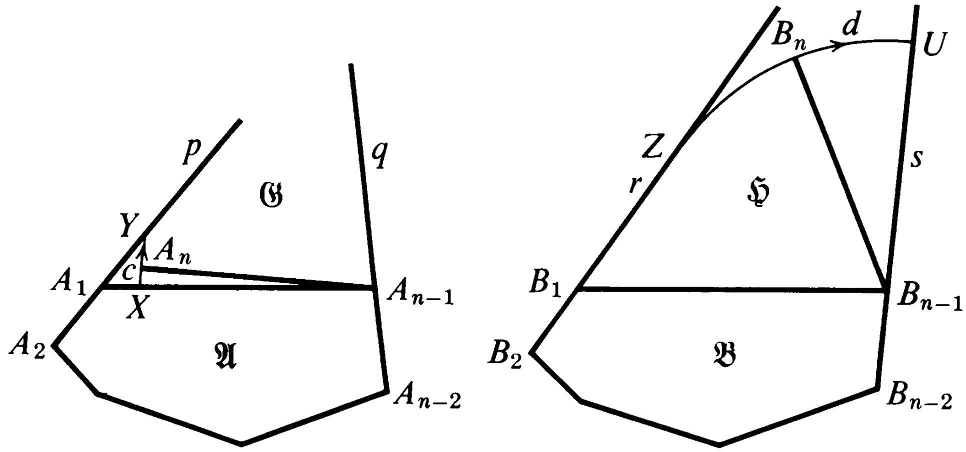


Fig. 93

If these two new polygons are convex, they fall under the first case (one of inequalities (2) is an equality), and so $A_1A_n' < B_1B_n'$. On the other hand, the two polygons $A_1A_2 \ldots A_n$ and $A_1A_2 \ldots A_n'$ fall under the first case (because the angles at vertices $A_2, A_3, \ldots, A_{n-2}$ coincide, and at vertex A_{n-1} the second polygon has the greater angle), that is, $A_1A_n < A_1A_n'$. For the very same reasons $B_1B_n' < B_1B_n$. Merely by combining the inequalities we have obtained, we produce the desired inequality $A_1A_n < B_1B_n$.

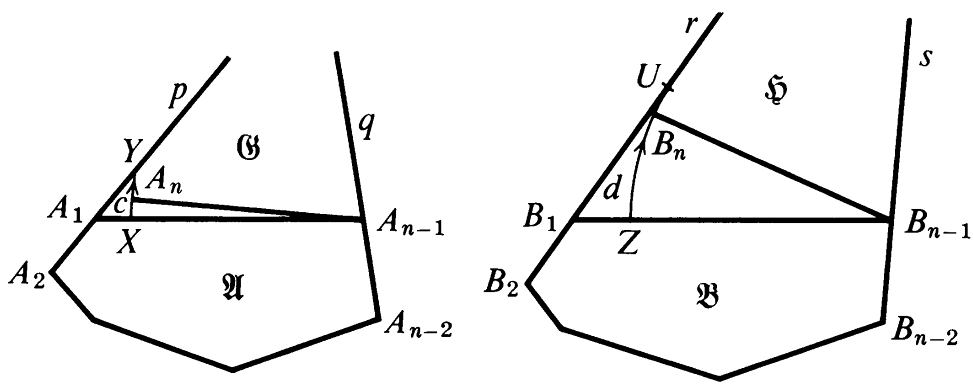
Now we must verify that the new polygons will be convex. Is it possible to turn sides $A_{n-1}A_n$ and $B_{n-1}B_n$ so that they form equal angles with sides $A_{n-2}A_{n-1}$ and $B_{n-2}B_{n-1}$ respectively, while preserving convexity?

Fig. 94



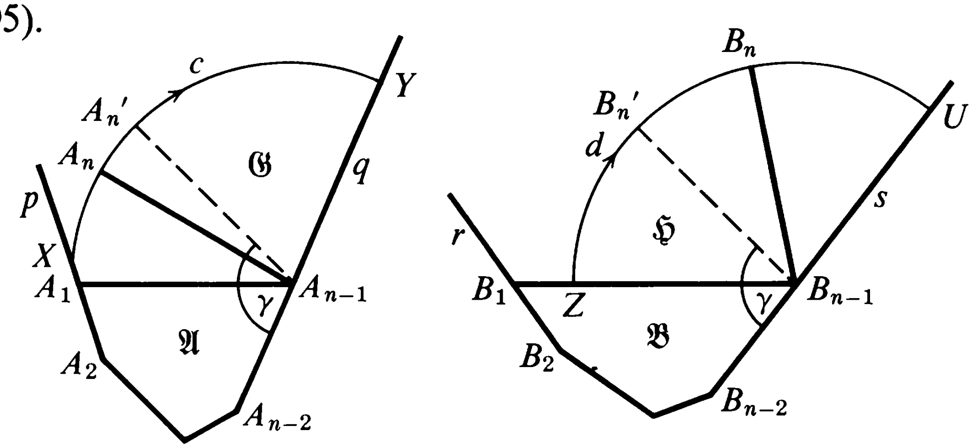
Let p and q denote the extensions of sides A_2A_1 and $A_{n-2}A_{n-1}$, and $\mathfrak{G}$, the part of the plane bounded by p, q, and A_1A_{n-1}. ($\mathfrak{G}$ may be finite, as in Figures 94 and 95, or infinite, as in Figures 93, 96, and 97.) Similarly, let r and s denote the extensions of B_1B_2 and $B_{n-2}B_{n-1}$, and $\mathfrak{H}$, the part of the plane bounded by r, s, and B_1B_{n-1}.

Fig. 95



Polygon $A_1A_2 \ldots A_{n-1}A_n'$ will be convex so long as $A_{n-1}A_n'$ lies in $\mathfrak{G}$, and polygon $B_1B_2 \ldots B_{n-1}B_n'$ will be convex when $B_{n-1}B_n'$ lies in $\mathfrak{H}$. Let us draw an arc $\widehat{XY}$ of a circle through point A_n with radius $A_{n-1}A_n$ and center A_{n-1}, lying in $\mathfrak{G}$, with X lying on A_1A_{n-1} (Fig. 93, 94, 95) or p (Fig. 96, 97) and Y lying on q (Fig. 93, 96, 97) or p (Fig. 94, 95).

Fig. 96

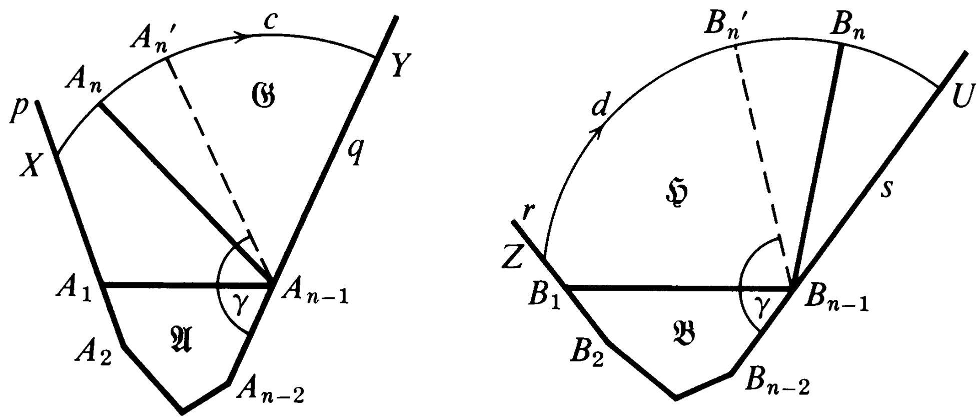


Polygon $A_1A_2 \ldots A_{n-1}A_n'$ will remain convex as A_n' moves along $\widehat{XY}$. Analogously we draw $\widehat{ZU}$ in $\mathfrak{H}$ with radius $B_{n-1}B_n$ and center at B_{n-1}, with Z lying on B_1B_{n-1} (Fig. 93, 95, 96) or r (Fig. 94, 97) and U lying on s (Figures 93, 94, 96, 97) or r (Fig. 95). Polygon $B_1B_2 \ldots B_{n-1}B_n'$ remains convex as B_n' runs along $\widehat{ZU}$.

As we remarked above, the lemma will be proved if there are points A_n' and B_n' on $\widehat{XY}$ and $\widehat{ZU}$ such that

$$\angle A_{n-2}A_{n-1}A_n' = \angle B_{n-2}B_{n-1}B_n'.$$

Fig. 97



We are given

$$\angle A_{n-2}A_{n-1}A_n < \angle B_{n-2}B_{n-1}B_n. \tag{7}$$

Also because polygon $A_1A_2 \ldots A_n$ is convex, point A_n coincides with neither X nor Y (Fig. 93–97), and so

$$\angle A_{n-2}A_{n-1}X < \angle A_{n-2}A_{n-1}A_n < \angle A_{n-2}A_{n-1}Y. \tag{8}$$

Analogously,

$$\angle B_{n-2}B_{n-1}Z < \angle B_{n-2}B_{n-1}B_n < \angle B_{n-2}B_{n-1}U. \tag{9}$$

There are two possible cases:

Case (i). $$\angle B_{n-2}B_{n-1}Z < \angle A_{n-2}A_{n-1}Y, \tag{10}$$

Case (ii). $$\angle B_{n-2}B_{n-1}Z \geq \angle A_{n-2}A_{n-1}Y. \tag{11}$$

In Case (i) (Fig. 93, 95, 96, 97, 98)[1] it is possible to find an angle γ that simultaneously satisfies the inequalities:

$$\left.\begin{matrix}\angle A_{n-2}A_{n-1}A_n \\ \angle B_{n-2}B_{n-1}Z\end{matrix}\right\} < \gamma < \left\{\begin{matrix}\angle B_{n-2}B_{n-1}B_n \\ \angle A_{n-2}A_{n-1}Y\end{matrix}\right.$$

[1] The case illustrated in Figures 95 and 98 may fall under Case (i), satisfying inequality (10), or Case (ii), satisfying inequality (11).

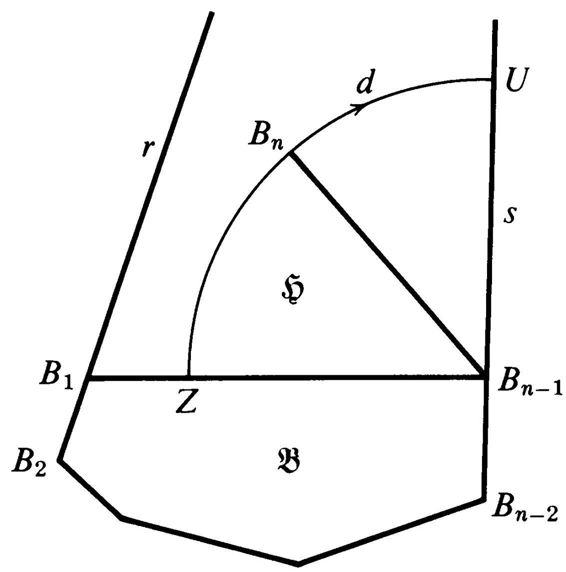
(since each angle on the right is greater than each angle on the left). Then there are points A_n' and B_n' on $\widehat{A_nY}$ and $\widehat{ZB_n}$ such that

$$\angle A_{n-2}A_{n-1}A_n' = \gamma = \angle B_{n-2}B_{n-1}B_n'$$

and the lemma is proved.

In Case (ii) point Y cannot possibly lie on q. (If it did, angle $A_{n-2}A_{n-1}Y$ would be equal to π and we would have Case (i).) Hence, Y lies on p (Fig. 94) and X lies on A_1A_{n-1}. But Z may lie either on r (Fig. 94) or on B_1B_{n-1} (Fig. 98).

Fig. 98



(a) Let Z lie on r. Consider the $(n-1)$-gons $A_2A_3 \ldots A_{n-1}Y$ and $B_2B_3 \ldots B_{n-1}Z$. They have the following pairs of equal sides:

$$A_2A_3 = B_2B_3, \quad \ldots, \quad A_{n-2}A_{n-1} = B_{n-2}B_{n-1},$$

and, finally,

$$A_{n-1}Y = A_{n-1}A_n = B_{n-1}B_n = B_{n-1}Z.$$

By inequalities (11) and (2), we have the angles of the second polygon at vertices $B_3, \ldots, B_{n-1}$ greater than or equal to the corresponding angles of the first polygon at vertices $A_3, \ldots, A_{n-1}$. Since we assumed the theorem to be true for $(n-1)$-gons, we have

$$B_2Z > A_2Y.$$

Since $B_2B_1 = A_2A_1$ from formula (1), we can substract them from the above inequality, obtaining

$$B_1Z > A_1Y. \tag{12}$$

Consider angle YA_1A_{n-1} (Fig. 94), which contains arc $\widehat{XY}$ with center A_{n-1} lying on side A_1A_{n-1} of the angle. In moving along this arc in the direction from X to Y, we obviously move away from A_1. Therefore,

$$A_1Y > A_1A_n. \tag{13}$$

On the other hand,

$$B_1Z < B_1B_n. \tag{14}$$

From inequalities (12), (13), and (14), we obtain the desired inequality:

$$B_1B_n > A_1A_n.$$

(*b*) The case for Z lying on B_1B_{n-1} remains (Fig. 98). The $(n-1)$-sided polygons $A_2A_3 \ldots A_{n-1}Y$ and $B_2B_3 \ldots B_{n-1}Z$ still satisfy all the above inequalities which precede $A_2Y < B_2Z$; so they satisfy it too. Since $A_2Y = A_1A_2 + A_1Y$ and $B_2Z < B_1B_2 + B_1Z$ (the triangle inequality for B_1B_2Z),

$$A_1A_2 + A_1Y < B_2Z < B_1B_2 + B_1Z.$$

Since $A_1A_2 = B_1B_2$, we subtract them from the above inequality to obtain

$$A_1Y < B_1Z.$$

But we already know that $A_1A_n < A_1Y$. Also, $B_1Z < B_1B_n$ (Fig. 98). Therefore

$$A_1A_n < A_1Y < B_1Z < B_1B_n.$$

Thus $A_1A_n < B_1B_n$ and Lemma 1 is completely proved. The remainder of Cauchy's proof is valid.

Supplement. We shall call two surfaces *isometric* if a topological correspondence can be established between them under which each line on one surface corresponds to a line on the other surface with the same length. Obviously, the surfaces of two polyhedra satisfying the conditions of Cauchy's theorem are isometric. The theorem states that they are congruent or symmetric.

Cauchy's theorem was generalized in 1941 by S. P. Olovyanshchikov, who proved:

Any convex surface isometric to the surface of a convex polyhedron is congruent or symmetric to it.

Starting in the late 19th century, several mathematicians showed convex isometric surfaces to be congruent or symmetric under these or other conditions. In 1949 A. V. Pogorelov proved the complete and final theorem about isometric surfaces:

Two arbitrary isometric convex surfaces are either congruent or symmetric.

23. ABSTRACT AND CONVEX POLYHEDRA; STEINITZ' THEOREM

DEFINITION. *The expression* "A is incident with B" *means* "*A contains B*" *or* "*B contains A.*"

In particular, if edge p forms part of the boundary of face a, then p is incident with a (and a is incident with p). Vertex A is incident with edge p (and p is incident with A) if point A is an end point of p. If A is one of the vertices of face a, then A is incident with a (and a is incident with A). Two elements of the same type (that is, two vertices, two edges, or two faces) are never incident.

The network made up of the vertices, edges, and faces of a convex polyhedron has the following properties:

- Ia. Each edge is incident with two and only two vertices.
- Ib. Each edge is incident with two and only two faces.
- IIa. There may be only one edge incident with both of two given vertices.
- IIb. There may be only one edge incident with both of two given faces.
- IIIa. Every vertex is incident with at least three faces.
- IIIb. Every face is incident with at least three vertices.

The concept of incidence may be extended to the elements of an arbitrary network on a surface. Each region in the network is incident with the lines which form parts of its boundary, and with the nodes lying on its boundary (and conversely). Each line of the network is incident with the nodes at its ends (and conversely).

Now let us agree to call the regions, lines, and nodes of a network its *faces, edges,* and *vertices,* respectively.

A *connected network* was defined in section 17. More precisely, we shall say that a network is *connected* if for any pair A, A' of its vertices there exists a sequence $a_1 = \text{A}, a_2, a_3, \ldots, a_n = A'$ of alter-

nating vertices and edges such that the adjacent pairs a_1 and a_2, a_2 and a_3, . . . , a_{n-1} and a_n are incident. The reader should be able to prove that any convex polyhedron is an example of a connected network.

DEFINITION. *A network on a convex surface will be called an* abstract polyhedron *if it is connected and satisfies conditions* Ia, Ib, IIa, IIb, IIIa, *and* IIIb.

DEFINITION. *Two polyhedra (ordinary or abstract) are said to be* equivalent *if it is possible to establish a one-to-one correspondence between their vertices, edges, and faces preserving incidence.*

In other words, each vertex, edge, or face of one polyhedron corresponds to a unique vertex, edge, or face, respectively, of the other polyhedron such that a pair of incident elements of one polyhedron correspond to a pair of incident elements of the other. For example, if some edge s of the first polyhedron is incident with face a and the corresponding edge and face of the second polyhedron are s_1 and a_1, then s_1 is incident with a_1. Thus, all tetrahedra are equivalent to each other; any parallelepiped and cube are equivalent.

The German mathematician Steinitz proved the following theorem, called the Fundamental Theorem of the Theory of Polyhedra.

STEINITZ' THEOREM. *For any abstract polyhedron there exists an equivalent convex polyhedron.*

Sometimes this is expressed thus: *Any abstract polyhedron may be realized as a convex polyhedron.*

Before proving this theorem, we shall give some necessary preliminary material. We shall consider the following *partitions* of a face a of a polyhedron.

Partition of type I: connect vertices A and B of face a by line AB, dividing a into two parts a_1 and a_2 (Fig. 99).

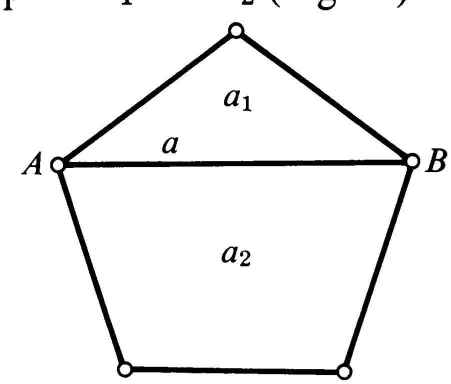


Fig. 99

Partition of type II: connect vertex A of face a with point B lying inside a side not incident with A, dividing a into two parts a_1 and a_2 (Fig. 100).

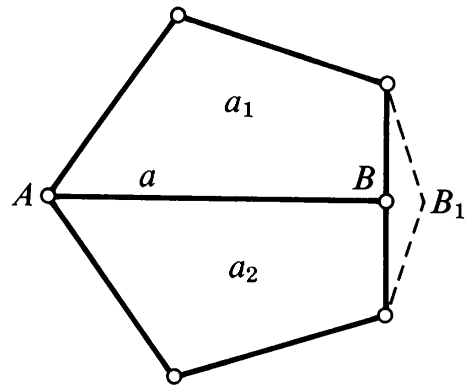


Fig. 100

Partition of type III: connect two points A and B lying inside two different sides of a, dividing a into two parts a_1 and a_2 (Fig. 101).

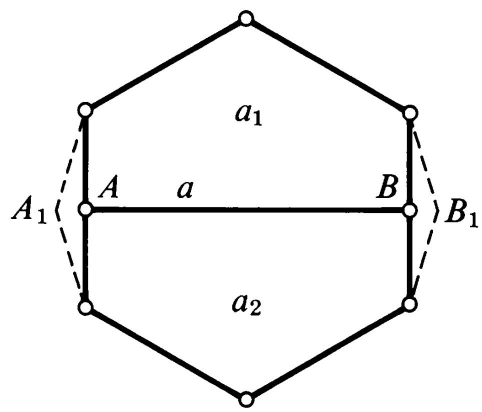


Fig. 101

If we subject some of the faces of an abstract polyhedron to partitions of types I, II, and III, then it is transformed into another abstract polyhedron. If the faces are plane polygons, then the lines AB are straight line segments.

LEMMA 1. *Any abstract polyhedron with n faces ($n \geq 5$) can be obtained from some abstract polyhedron with $n - 1$ faces by means of a partition of type I, II, or III. Consequently, any abstract polyhedron with n faces can be obtained from an abstract tetrahedron by applying partitions of types I, II, and III.*

Proof. Suppose that we have an abstract polyhedron P_n. Choose an arbitrary vertex A of this polyhedron. At least three edges AB, AC, and AD are incident with it. It is possible to connect points B, C, and D by broken lines BC, CD, and DB composed of edges of the polyhedron other than AB, AC, and AD. The lines AB, AC, AD, BC, CD, DB (the heavy lines in Fig. 102) divide our polyhedron

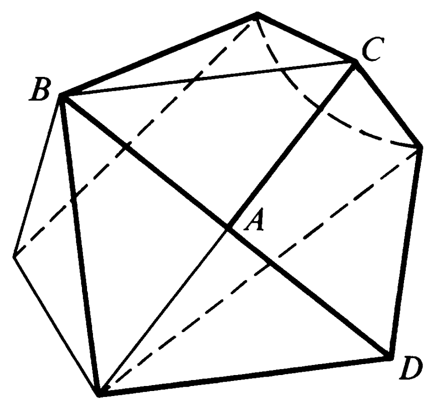


Fig. 102

into 4 parts, which may be considered as the "faces" of an abstract tetrahedron (4-hedron) with vertices A, B, C, D and edges AB, AC, AD, BC, CD, DB.

We now draw all the other lines composed of edges of the polyhedron. In doing so we make a sequence of partitions of the first, second, and third kinds, and our 4-hedron changes into a 5-hedron, then a 6-hedron, . . . , until finally we obtain our initial n-hedron. Thus, Lemma 1 is proved.

Now we shall begin the proof of Steinitz' theorem. It will be proved by induction.

Proof of Steinitz' theorem. For 4-hedra it is true that every abstract 4-hedron is equivalent to a tetrahedron. In fact, every face of a 4-hedron has 3 sides. (It cannot have fewer than three sides by the definition of an abstract polyhedron, and it cannot have more than three sides because it cannot border more than three faces.) Hence, we have 4 triangles. The only polyhedron that can be made with them is a tetrahedron.

We now assume that the theorem is proved for all $(n - 1)$-hedra, and we shall show that it follows that it is valid for n-hedra $(n \geq 5)$.

Consider an abstract n-hedron P_n. It can be obtained from some abstract $(n - 1)$-hedron by a partition of the first, second, or third kind by Lemma 1. According to our assumption, the $(n - 1)$-hedron may be realized as a convex $(n - 1)$-hedron P_{n-1}. P_n may be obtained by a partition of some face a of P_{n-1}. If the partition is of type I, A and B are vertices of P_{n-1} (Fig. 99). If it is of type II, one of the ends, for example A, of AB is a vertex of a, and the other end B lies inside an edge (Fig. 100). And if the partition is of type III, A and B both lie inside edges of P_{n-1} (Fig. 101).

From this partition we obtain first the n-hedron $\overline{P}_n$, equivalent to P_n. But $\overline{P}_n$ is not strictly convex, two of its faces, a_1 and a_2, obtained by partitioning face a, lie in the same plane. Let us see if it is possible to transform $\overline{P}_n$ into a strictly convex polyhedron P_n' equivalent with it. If so, then P_n' would be a realization of P_n as a convex polyhedron. At first glance it appears simple. In the case of a partition of type I, we rotate one of the parts into which a is divided. For example, we rotate a_1 around AB in a sufficiently small angle, holding a_2 fixed, so that the dihedral angle between a_1 and a_2 is less than $180°$ and all vertices not incident with a_1 turn out to be located on one side of a_1, so that $\overline{P}_n$ become a strictly convex polyhedron. In case the partition is of type II, the addition of the new vertex B changed the boundary of a into a not strictly convex polygon. (The two sides into which B divides one of the sides of a lie on one straight line.) We move vertex B to the position B_1 (Fig. 100) so that a becomes a strictly convex polygon, and then continue as in the above case. For type III, we begin analogously by moving both new vertices A and B so that a (with the addition of the new vertices) becomes a strictly convex polygon, and then continue as in the first case.

But here we encounter difficulties. We do not know how legitimate it is to displace these faces and vertices, even though the displacement can be made as small as you please. We shall explain in greater detail.

Consider the set E of all vertices $A_1, \ldots, A_m$ of a polyhedron P and the set F of all its faces, $a_1, a_2, \ldots, a_n$. Let e_i be the number of faces of P incident with vertex A_i, and let f_j be the number of vertices incident with face a_j. The numbers e_i and f_j will be called

the numbers of incidences of the elements (vertex and face) A_i and a_j, respectively.

Let all $e_i = 3$; that is, all vertices belong to three faces (for example, consider a cube). Now let us change the positions of the faces of the polyhedron, that is, move the planes of these faces. For sufficiently small changes in the planes, the vertices, the points of intersection of sets of three faces, are displaced as little as you please. If only 3 faces intersect in each vertex, then after a sufficiently small displacement there will still only be 3 faces intersecting in each vertex. Also, after a sufficiently small motion, the property of convexity, which may be expressed by saying that all vertices not incident with any given face lie on the same side of the face, is preserved. So, a small motion of the faces transforms a convex polyhedron into another equivalent, almost congruent, convex polyhedron.

Now suppose that at least one of the numbers of incidences $e_i > 3$, that is, that some vertex A_i is incident with 4 or more faces (for example, consider a regular octahedron). If 4 or more faces intersect in vertex A_i, then it is possible to move one of these faces an arbitrarily small distance so that they cease to intersect at one point.

Analogously, if all $f_j = 3$ (all faces are triangles), then by a sufficiently small displacement of the vertices, all the triangular faces are transformed into nearly similar triangular faces and the polyhedron is still convex. If one of the numbers $f_j > 3$ (if some face a_j is incident with 4 or more vertices, as on a cube), then there exist arbitrarily small displacements of the vertices incident with face a_j (for $f_j > 3$) after which these f_j vertices no longer lie on one plane. Thus, our problem is not simple to solve. We need some further theory.

Consider the set $E + F$ of all vertices and all faces. It has $m + n$ elements

$$\alpha_1, \alpha_2, \ldots, \alpha_{m+n}. \tag{1}$$

Each of $\alpha_1, \alpha_2, \ldots$ is either a vertex or a face.

DEFINITION. *Sequence* (1) *is a* proper sequence *if for each* α_i, $1 \leq i \leq m + n$, *among the preceding elements* $\alpha_1, \alpha_2, \ldots, \alpha_{i-1}$ *there are not more than* 3 *incident with* α_i.

In other words, if α_i is a face of P, then among $\alpha_1, \alpha_2, \ldots, \alpha_{i-1}$ there are not more than 3 vertices incident with it; if α_i is a vertex of P, then among $\alpha_1, \alpha_2, \ldots, \alpha_{i-1}$ there are not more than 3 faces

incident with it. (We remind the reader that a pair of vertices or a pair of faces are never incident.) Notice that the sequence α_1, α_2, α_3, α_4 is always a proper sequence.

For example, let $\alpha_1, \alpha_2, \ldots, \alpha_6$ be the faces and $\alpha_7, \alpha_8, \ldots, \alpha_{14}$ the vertices of a cube. The sequence

$$\alpha_1, \alpha_2, \ldots, \alpha_{14}$$

is a proper sequence. For any face α_i ($i \leq 6$), none of the preceding α are incident with it because they are all faces. For any vertex α_j ($j \geq 7$), there are 3 incident elements preceding it. They are the three faces incident with this vertex. On the other hand, if these elements are arranged with the vertices in the first 8 places of the sequence, followed by the 6 faces, then we do not have a proper sequence. Each face here is preceded by 4 vertices incident with it, the vertices lying on its boundary.

Suppose that we have a proper sequence

$$\alpha_1, \alpha_2, \ldots, \alpha_{m+n}.$$

Let us displace the element α_1 (which may be either a vertex or a face) by an arbitrary amount. Then we shall displace the following elements $\alpha_2, \alpha_3, \alpha_4, \ldots$ in order. Suppose that the elements α_1, α_2, $\ldots$, α_{i-1} have already been displaced. If there are three elements among them incident with α_i, then the displacement of α_i is already determined. (If α_i is a face, then the displacement of three vertices incident with it defines the displacement of the face.[1] If α_i is a vertex, then the displacement of three faces intersecting in it defines the displacement of α_i.) The displacement of α_i can be made arbitrarily small by making the displacements of the preceding elements sufficiently small.

Among $\alpha_1, \alpha_2, \ldots, \alpha_{i-1}$ there may be only one or two elements incident with α_i. Then there will be infinitely many ways of displacing α_i so that it is still incident with the one or two elements. It is still possible to make the displacement of α_i as small as we please by making the displacements of the preceding elements sufficiently small. If none of $\alpha_1, \alpha_2, \ldots, \alpha_{i-1}$ are incident with α_i, then α_i may be displaced any way we like.

Thus, after any sufficiently small displacement of α_1, we can displace the other elements one after the other so that all incidences are preserved; that is, the polyhedron is transformed into an equiv-

[1] Only if no pair of these vertices merges. We must make all displacements small enough to prevent distinct elements from merging.

alent one. We shall also require the displacements to be small enough to preserve the convexity of the polyhedron.

Now we return to polyhedron $\overline{P}_n$. We assume that its faces and vertices may be arranged in a proper sequence such that for a partition of type I the first two places are occupied by the elements a_1 and a_2:

$$\alpha_1 = a_1, \quad \alpha_2 = a_2, \quad \alpha_3, \quad \alpha_4, \quad \ldots, \quad \alpha_{m+n}.$$

For a partition of type II we assume that the first three places are filled by B, a_1, a_2:

$$\alpha_1 = B, \quad \alpha_2 = a_1, \quad \alpha_3 = a_2, \quad \alpha_4, \quad \ldots, \quad \alpha_{m+n}.$$

For a partition of type III we assume that the first four places are occupied by A, B, a_1, a_2:

$$\alpha_1 = A, \quad \alpha_2 = B, \quad \alpha_3 = a_1, \quad \alpha_4 = a_2, \quad \alpha_5, \quad \ldots, \quad \alpha_{m+n}.$$

In case the partition is of type I, we first rotate one of faces a_1 or a_2 around AB through an angle so small that all vertices not incident with either of these faces stay on the same side of them. Then we displace the remaining elements as described above. If the initial displacement is sufficiently small, $\overline{P}_n$ is transformed into an equivalent strictly convex polyhedron P_n'.

If the partition is of type II (or III), we first displace the new vertex B (or the two new vertices A and B) in the plane of the partitioned face a so that a becomes a strictly convex polygon, and continue as for a partition of type I. Thus we transform $\overline{P}_n$ into an equivalent strictly convex polyhedron P_n'.

Proof of Steinitz' theorem (continued). To complete the proof of the theorem, we must show that it is always possible to arrange the elements (vertices and faces) of $E + F$ into a proper sequence with the desired elements in the first few places, as is described above.

Suppose that we are given some network S on the surface of a sphere or some other convex solid. Let E denote the set of all vertices of S, F the set of all its faces, and $E + F$ the set of all vertices and faces of the network S. Also, let E^* denote a subset of E, that is, some set of vertices in E, but not necessarily all of them, and let F^* denote a subset of F (some set of faces). The vertices in E^* and the faces in F^* will be the elements of $E^* + F^*$. Any pair of incident elements of $E^* + F^*$ consists of one vertex in E^* and one face in F^*.

Let m be the number of vertices in E^* and n the number of faces in F^*. After arranging the vertices in E^* in some order, let e_i be the number of faces in F^* incident with the ith vertex of E^*. Analogously, we arrange the faces in F^* in some order and let f_j denote the number of vertices in E^* incident with jth face in F^*. The total number of incident pairs may be expressed as the sum of all e_i or as the sum of all f_j, and so we have the equation:

$$\sum_{i=1}^{m} e_i = \sum_{j=1}^{n} f_j. \tag{2}$$

LEMMA 2. *For any set $E^* + F^*$ consisting of more than two elements, we have:*

$$\sum_{i=1}^{m} (4 - e_i) + \sum_{j=1}^{n} (4 - f_j) \geq 8. \tag{3}$$

Proof. There will be five parts.

(i) By equation (2), the left side of inequality (3) may be written in the form

$$\sum_{i=1}^{m} (4 - e_i) + \sum_{j=1}^{n} (4 - f_j) = 4(m + n) - \left(\sum_{i=1}^{m} e_i + \sum_{j=1}^{n} f_j \right)$$

$$= 4(m + n) - 2 \sum_{i=1}^{m} e_i = 4(m + n) - 2 \sum_{j=1}^{n} f_j.$$

We shall denote this expression by the symbol

$$\sum (E^* + F^*) = 4(m + n) - 2 \sum_{j=1}^{n} f_j. \tag{4}$$

This sum is equal to four times the number of elements of $E^* + F^*$ minus twice the number of incident pairs of elements. If $E^* + F^*$ consists of a single element, then there are no incident pairs and consequently,

$$\Sigma\,(E^* + F^*) = 4. \tag{5}$$

If $E^* + F^*$ consists of two elements, then there can be only one pair of incident elements, and $\Sigma\,(E^* + F^*)$ equals 6 or 8, depending on whether the two elements are incident or not. If $m + n = 3$, then $E^* + F^*$ can have not more than two pairs of incident elements. (Either one face and two vertices incident with it, or conversely.) Consequently, $\Sigma\,(E^* + F^*)$ may equal 8, 10, or 12. So, the theorem is true for $m + n = 3$.

(ii) If $E^* + F^* = E + F$ (that is, $E^* + F^*$ consists of all the vertices and faces of the original network), then the theorem follows immediately from (5) in section 19:

$$m - 2 \geq l - n$$
$$m + n - l \geq 2$$

where m, n, and l are the numbers of vertices, faces, and edges. In fact, f_j, the number of vertices in E incident with the jth face of the polyhedron, equals the number of edges of this polygon, and $\sum_{j=1}^{n} f_j$ is twice the number, l, of edges in the network (because each edge is counted twice):

$$\sum_{j=1}^{n} f_j = 2l,$$

$$\Sigma\,(E + F) = 4(m + n) - 2\,\Sigma f_j = 4(m + n - l) \geq 8.$$

(iii) Notice that if we remove some face a_j from $E^* + F^*$, then $\Sigma\,(E^* + F^*)$ increases by $2f_j - 4$. In fact, $m + n$ decreases by 1, and Σf_j decreases by f_j; consequently, $4(m + n) - 2\,\Sigma f_j$ increases by $2f_j - 4$.

If $f_j \geq 2$, that is, $2f_j - 4 \geq 0$, then the removal of face a_j does not decrease $\Sigma\,(E^* + F^*)$.

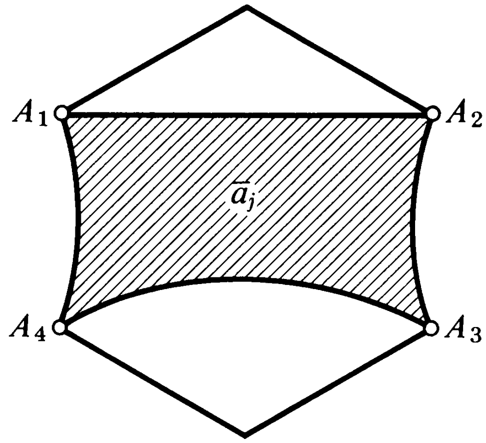
If $f_j < 2$ ($f_j = 0$ or 1), then $2f_j - 4 < 0$ and the removal of face a_j decreases $\Sigma\,(E^* + F^*)$. Analogously, this sum is decreased by the removal of a vertex for which $e_i = 0$ or 1, but is not decreased by the removal of a vertex for which $e_i \geq 2$.

(iv) Now suppose that for $E^* + F^*$ all $e_i \geq 2$ and all $f_j \geq 2$. Consider any face a_j in F^* that is incident with f_j vertices in E^*, which we denote by $A_1, A_2, \ldots, A_{f_j}$ (A_1, A_2, A_3, A_4 in Fig. 103), arranged in cyclic order on the boundary of a_j.

Now we draw lines inside a_j to connect these vertices in order, obtaining a polygon $\bar{a}_j$ (the shaded area in Fig. 103) incident only with the vertices of a_j which belong to E^*. These are the only vertices of $\bar{a}_j$, although a_j may have vertices in E which are not in E^*. Let $\bar{F}^*$ denote the set of new polygons obtained by performing this construction in all faces in F^*. Then we have

$$\Sigma\,(E^* + F^*) = \Sigma\,(E^* + \bar{F}^*),$$

Fig. 103



because each polygon $\bar{a}_j$ in $\bar{F}^*$ is incident with exactly the same vertices in E^* as the corresponding a_j in F^*. The vertices and edges of the $\bar{a}_j$ in $\bar{F}^*$ form a new network on our convex surface. Its vertices are the elements of E^* and its faces are the elements of $\bar{F}^* + \bar{\bar{F}}^*$ where $\bar{\bar{F}}^*$ denotes the faces not in $\bar{F}^*$. Part (ii) tells us that for this new network,

$$\Sigma\,(E^* + \bar{F}^* + \bar{\bar{F}}^*) \geq 8.$$

In order to obtain $E^* + \bar{F}^*$ from $E^* + \bar{F}^* + \bar{\bar{F}}^*$, we must remove the faces in $\bar{\bar{F}}^*$.

Each face in $\bar{F}^* + \bar{\bar{F}}^*$ is incident with more than two vertices of E^*. From part (iii) above, removal of a face will not decrease $\Sigma\,(E^* + \bar{F}^* + \bar{\bar{F}}^*)$. Since this sum is ≥ 8, by removing the faces in $\bar{\bar{F}}^*$ we obtain

$$\Sigma\,(E^* + \bar{F}^*) \geq 8.$$

(v) One case remains: $E^* + F^*$ has $m + n > 3$ elements, and some of the numbers e_i or f_j are less than 2. Remove one of the elements for which e_i or f_j is less than 2. We obtain a new system. If it has more than 3 elements and e_i or f_j is less than 2 for some element, remove that element, obtaining another system. Repeating this process, we finally obtain a system with only 3 elements, or a system for which all e_i and $f_j \geq 2$. In either case the sum (4) ≥ 8. With the removal of elements for which e_i or $f_j < 2$, this sum decreases. Hence, the sum for the original system must have been larger, that is, $\Sigma\,(E^* + F^*) \geq 8$, and Lemma 2 is proved.

LEMMA 3. *The set $E + F$ of vertices and faces of any network may be arranged in a proper sequence* (p. 86).

Proof. Among the elements of any system $E^* + F^*$ there is at least one for which e_i or $f_j \leq 3$, by virtue of the inequality

$$\Sigma (4 - e_i) + \Sigma (4 - f_j) \geq 8.$$

(Some of the terms $4 - e_i$ and $4 - f_j$ must be positive because their sum is positive. Hence, some e_i or f_j must be ≤ 3.) Therefore, among the $m + n$ elements of $E + F$ there is one which is incident with not more than three of the remaining $m + n - 1$ elements. Denote it by α_{m+n}. Among the remaining $m + n - 1$ there is at least one incident with not more than three of the remaining $m + n - 2$. Denote it by α_{m+n-1}. Continuing, we form a sequence of elements $\alpha_{m+n}, \alpha_{m+n-1}, \ldots, \alpha_r, \ldots$ in which each α_r was chosen so that it is incident with not more than three of the remaining $r - 1$ elements. Arranging this sequence in the opposite order, we obtain the proper sequence

$$\alpha_1, \alpha_2, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_{m+n-1}, \alpha_{m+n},$$

and Lemma 3 is proved.

Notice that Lemma 3 holds for any system $E^* + F^*$, with the same proof. If the system $E^* + F^*$ is divided into two subsystems, $E_1 + F_1$ and $E_2 + F_2$, containing n_1 and n_2 elements respectively, we can use the following sharper statement.

LEMMA 4. *If*

$$\Sigma (E_1 + F_1) \leq 8, \tag{6}$$

then the elements of $E^* + F^*$ can be arranged in a proper sequence in which the first n_1 places are filled by the elements of $E_1 + F_1$.

Proof. There are two possible cases.

Case 1. There is at least one element of $E_2 + F_2$ incident with some element of $E_1 + F_1$.

We must agree on our notation. As before, for each element of $E^* + F^*$, e_i or f_j will denote the number of elements of $E^* + F^*$ incident with it. Also, for a given element of $E_1 + F_1$, e_i' or f_j' will denote the number of elements of $E_1 + F_1$ incident with it. Notice that at least one of the numbers e_i' or f_j' is less than the corresponding number e_i or f_j, because at least one element of $E_1 + F_1$ is

incident with some element of $E_2 + F_2$. The symbols Σ_1 and Σ_2 will indicate sums taken over the elements of $E_1 + F_1$ and $E_2 + F_2$ respectively, and Σ will indicate summation over all elements of $E^* + F^*$.

We have:

$$\begin{aligned}\Sigma\,(E^* + F^*) &= \Sigma\,(4 - e_i) + \Sigma\,(4 - f_j) \\ &= [\Sigma_1 (4 - e_i) + \Sigma_1 (4 - f_j)] + [\Sigma_2 (4 - e_i) + \Sigma_2 (4 - f_j)].\end{aligned}$$

As we saw above, one of the numbers e_i or f_j for some element of $E_1 + F_1$ is greater than the corresponding e_i' or f_j'. Therefore

$$\Sigma_1 (4 - e_i) + \Sigma_1 (4 - f_j) < \Sigma_1 (4 - e_i') + \Sigma_1 (4 - f_j').$$

But the expression on the right side of this inequality is identical with $\Sigma\,(E_1 + F_1)$, which is ≤ 8, by the conditions of the theorem. Hence

$$\Sigma_1 (4 - e_i) + \Sigma_1 (4 - f_j) \leq 7.$$

Since $\Sigma\,(E + F) = \Sigma\,(4 - e_i) + \Sigma\,(4 - f_j) \geq 8$, we have

$$\begin{aligned}\Sigma_2 (4 - e_i) + \Sigma_2 (4 - f_j) &= [\Sigma\,(4 - e_i) + \Sigma\,(4 - f_j)] \\ &\quad - [\Sigma_1 (4 - e_i) + \Sigma_1 (4 - f_j)] \\ &\geq 8 - 7 > 0.\end{aligned}$$

Consequently, at least one term of $\Sigma_2 (4 - e_i) + \Sigma_2 (4 - f_j)$ is positive; that is, for at least one of the elements of $E_2 + F_2$, the number e_i or f_j of elements of $E^* + F^*$ incident with it is less than or equal to 3.

Case 2. No elements of $E_2 + F_2$ are incident with elements of $E_1 + F_1$.

Applying the reasoning used in the proof of the preceding lemma to $E_2 + F_2$, we see that there must be an element of $E_2 + F_2$ which is incident with not more than three elements of $E_2 + F_2$. But since this element is incident with no element of $E_1 + F_1$, it is incident with not more than three elements of $E^* + F^*$.

Hence, in either Case 1 or Case 2, we can choose an element of $E_2 + F_2$ incident with not more than three elements of $E^* + F^*$. Denote this element by $a_{n_1+n_2}$. Now remove $a_{n_1+n_2}$ from $E_2 + F_2$ and from $E^* + F^*$. The above remarks hold for the resulting sets, so we can choose a second element $a_{n_1+n_2-1}$ of $E_2 + F_2$ which is incident with not more than three of the remaining elements of

$E^* + F^*$. Repeating this process until $E_2 + F_2$ is exhausted, we arrange the elements of $E_2 + F_2$ in a sequence $a_{n_1+n_2}, a_{n_1+n_2-1}, \ldots, a_{n_1+1}$, in which each element $a_{n_1+n_2-i}$ is incident with not more than three remaining elements of $E^* + F^*$ (that is, elements of $E^* + F^*$ other than $a_{n_1+n_2}, a_{n_1+n_2-1}, \ldots, a_{n_1+n_2-i}$).

By Lemma 3 we can arrange the elements of $E_1 + F_1$ in a proper sequence

$$a_1, a_2, \ldots, a_{n_1}.$$

Combining this with the elements of $E_2 + F_2$ arranged in the reverse of the order in which they were chosen, we obtain the desired proper sequence

$$a_1, a_2, \ldots, a_{n_1}, a_{n_1+1}, \ldots, a_{n_1+n_2},$$

in which the first n_1 places are taken by the elements of $E_1 + F_1$, and the last n_2 places are filled by the elements of $E_2 + F_2$. Thus, Lemma 4 is proved.

Notice that Lemma 4 includes the important special case for which $E^* = E$ and $F^* = F$.

EXAMPLES:

1. For the first two elements of a proper sequence we may choose any two faces a_1 and a_2 in F. In fact, if we denote the pair a_1, a_2 by $E_1 + F_1$ (actually, a_1 and a_2 belong to F_1, and E_1 is empty), then we have:

$$\Sigma\,(E_1 + F_1) = 8.$$

2. If we have two faces a_1, a_2, and a vertex B incident with both of them, then we can find a proper sequence with B, a_1, a_2 as its first three elements. In fact, if $E_1 + F_1$ consists of B, a_1, and a_2, then

$$\Sigma\,(E_1 + F_1) = 8.$$

3. If we have two vertices, A, B, and two faces a_1, a_2 incident with them, then there is a proper sequence beginning with A, B, a_1, a_2. In fact, if $E_1 + F_1$ denotes the set of these four elements, then

$$\Sigma\,(E_1 + F_1) = 8.$$

Proof of Steinitz' theorem (concluded). Now we can complete the proof of the theorem of Steinitz. $\overline{P}_k$ was obtained by a partition of type I, II, or III. Arrange the vertices and faces of $\overline{P}_k$ in a proper sequence. If the partition is type I, then the sequence begins with the elements a_1 and a_2; for type II, the sequence begins with B, a_1, and a_2; for type III, the sequence begins with A, B, a_1, a_2. This completes the proof of the theorem.

A refinement. We have defined *abstract polyhedron* to mean a connected network *on a convex surface* (such as the surface of a sphere) which satisfies conditions Ia–IIIb (page 81). We can obtain a greater degree of abstraction by eliminating the reference to a convex surface. First we must state one additional condition which holds for networks on surfaces:

IV. A face is incident with a vertex if and only if there is an edge incident with both.

Now we define a *completely abstract polyhedron* as a set of elements called "vertices," "edges," and "faces" with an incidence relation satisfying conditions Ia–IV, and for which Euler's condition

$$m + n - l = 2$$

holds. (m, n, and l are the numbers of vertices, faces, and edges, respectively.) It has been proved that any such polyhedron may be realized as a connected network on a sphere. Combining this result with the theorem just proved, we obtain the following form of the theorem of Steinitz.

Any completely abstract polyhedron can be realized as a convex polyhedron.

24. DEVELOPMENT OF A CONVEX POLYHEDRON; ALEKSANDROV'S THEOREM

If we cut a polyhedron R apart along its edges, we obtain a system of polygons $P_1, P_2, \ldots, P_n$. Each edge, since it borders two faces, will be considered as a side of both the corresponding polygons. It will be denoted by the same symbol a in both polygons. Similarly, a common vertex A of k faces ($k \geq 3$) will be considered as a common vertex of the k corresponding polygons and denoted by the same symbol A in all of them. Such a system of polygons

with common corresponding sides and vertices is called a *development* of the original polyhedron R. If we glue these polygons together along their common sides, we recover the polyhedron. By construction and definition we have the following.

The development of a convex polyhedron *is a system of polygons satisfying the following conditions:*

(1) *Each side is common to exactly two polygons; each vertex is common to at least three polygons.*

(2) *The numbers n of polygons, m of distinct vertices, and l of distinct sides satisfy the Euler relation*

$$m + n - l = 2.$$

(3) *Any two polygons Q and Q' may be connected by some sequence of polygons $Q = Q_0, Q_1, \ldots, Q_i = Q'$ for which Q_0 and Q_1, Q_1 and $Q_2, \ldots, Q_{i-1}$ and Q_i have common sides.*

(4) *Common sides of two polygons have equal lengths.*

(5) *The sum of the angles at a common vertex of several polygons is less than* 2π (because the sum of the plane angles of a convex polyhedral angle is less than 2π).

In his monograph *Convex Polyhedra,* A. D. Aleksandrov proved the following remarkable converse theorem:

ALEKSANDROV'S THEOREM. *A system of polygons satisfying conditions* (1)–(5) *above is a development of some convex polyhedron.*

Notice the connections between this theorem and Cauchy's theorem of section 20. Aleksandrov's theorem proves that there exists a convex polyhedron with a given development, and Cauchy's theorem proves that the polyhedron is unique (up to congruence and symmetry).

4. Linear Systems of Convex Figures

25. LINEAR OPERATIONS ON POINTS

In this chapter we assume that the reader is familiar with elementary analytic geometry and analysis.

Linear operations may be performed on vectors; that is, vectors may be added to vectors, and vectors may be multiplied by real numbers (scalars). We shall show how these operations may also be performed on convex solids. Let us consider each point q of the plane or of space as the end point of a vector issuing from some fixed point O, called the origin. The letters $\mathfrak{Q}$, $\mathfrak{A}$, $\mathfrak{B}$ will denote figures, O, A, B, C, p, q, r will denote points; l will denote a line; and s, x, y will denote real numbers. Also, subscripts and primes will be used when needed.

If a vector $\overrightarrow{Or}$ is the *sum* of vectors $\overrightarrow{Oq}$ and $\overrightarrow{Oq_1}$, that is,

$$\overrightarrow{Or} = \overrightarrow{Oq} + \overrightarrow{Oq_1},$$

then we shall call its end point, r, the *sum* of points q and q_1 (Fig. 104), and write:

$$r = q + q_1.$$

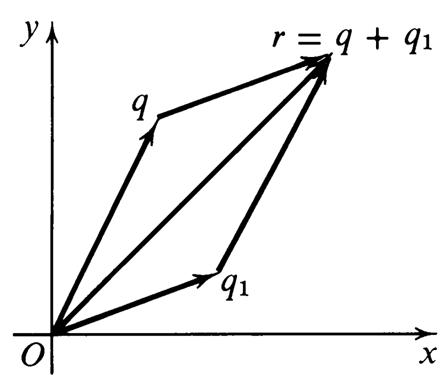


Fig. 104

The *product* sq, where q is a point and s is a real number, is defined analogously. Namely, $r = sq$ is the end point of the vector

$$\overrightarrow{Or} = s\overrightarrow{Oq}.$$

From these definitions, we have:

The line segment connecting two points q and q_1 (Fig. 105) *is the set of all points q_s defined by the formula*

$$q_s = sq + (1 - s)q_1 \quad (0 \leq s \leq 1), \tag{1}$$

or, with $s_1 = 1 - s$,

$$q_s = sq + s_1 q_1 \quad (s + s_1 = 1;\ s,\ s_1 \geq 0). \tag{1'}$$

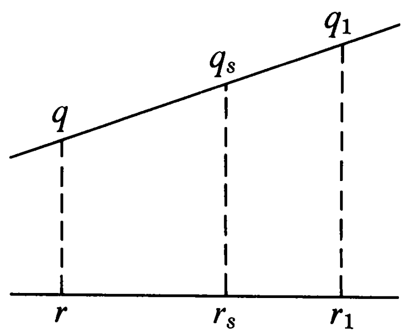


Fig. 105

Point q_s divides segment qq_1 in the ratio $(1 - s)/s$ or s_1/s. If $s = s_1 = \frac{1}{2}$, then $q_s = q_{\frac{1}{2}}$ is the midpoint of segment qq_1.

Let r, r_1, and r_s be the projections of points q, q_1, and q_s on some straight line or plane (Fig. 105). Then point r_s lies in segment rr_1 and divides it in the same ratio as q_s divides qq_1, namely $(1 - s)/s$. Hence,

$$r_s = sr + s_1 r_1 \quad (s + s_1 = 1;\ s,\ s_1 \geq 0). \tag{2}$$

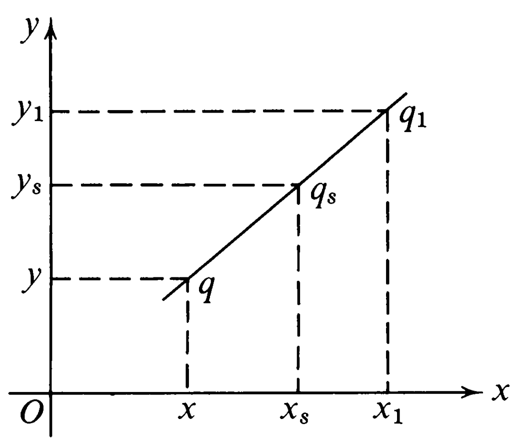


Fig. 106

If points q and q_1 (Fig. 106) have the coordinates (x, y) and (x_1, y_1) in the plane, then point q_s on segment qq_1 has the coordinates (x_s, y_s) given by

$$\left.\begin{aligned} x_s &= sx + s_1x_1 \\ y_s &= sy + s_1y_1 \end{aligned}\right\} \quad (s + s_1 = 1 \text{ and } s, s_1 \geq 0). \tag{3}$$

This assertion is a consequence of equation (2). If we project segment qq_1 on the coordinate axes Ox and Oy, then each point of the segment is projected onto the points corresponding to its abscissa and ordinate.

Suppose that we are given some function f. Its graph is the graph of the equation

$$y = f(x).$$

DEFINITION. *A function is called a* concave function *if its graph either lies above or coincides with any chord AB connecting two of its points* (Fig. 107).

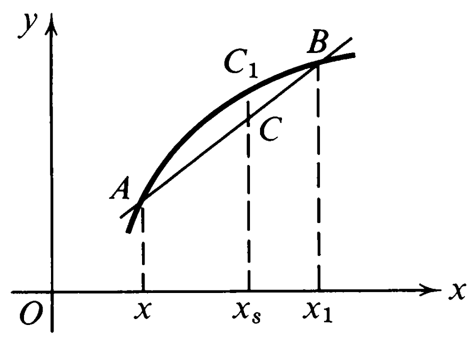


Fig. 107

In Figure 107, let x and x_1 be the abscissas, $y = f(x)$ and $y_1 = f(x_1)$ be the ordinates of points A and B. Each abscissa between x and x_1 may be given in the form

$$x_s = sx + s_1x_1 \quad (s + s_1 = 1;\ s, s_1 \geq 0). \tag{4}$$

Point C on chord AB with abscissa x_s has ordinate

$$y_s = sy + s_1y_1 = sf(x) + s_1f(x_1).$$

Point C_1 on the graph of $y = f(x)$ with the same abscissa x_s has ordinate $f(x_s)$. Since point C_1 either lies above C or coincides with it, its ordinate is greater than or equal to the ordinate of C:

$$f(x_s) \geq sf(x) + s_1f(x_1). \tag{5}$$

Hence, we could give the following equivalent definition of the concavity of f: A function f is *concave* if inequality (5) holds for all pairs of numbers x and x_1, and for all x_s between them given in equation (4).

26. LINEAR OPERATIONS ON FIGURES; "MIXING" FIGURES

DEFINITION. *Suppose that we are given two figures (or two solids) $\mathfrak{A}$ and $\mathfrak{B}$. Then their* sum, $\mathfrak{A} + \mathfrak{B}$, *will be the set of all points of the form $p + q$ where p belongs to $\mathfrak{A}$ and q belongs to $\mathfrak{B}$.*

For example, if $\mathfrak{A}$ is a segment of the x-axis with length 1 and with the origin as its left end point, and $\mathfrak{B}$ is a segment of the y-axis with length 1 and with the origin as its lower end point, then $\mathfrak{A} + \mathfrak{B}$ is a square region with sides $\mathfrak{A}$ and $\mathfrak{B}$. Also, if $\mathfrak{B}$ consists of only one point q_0, then $\mathfrak{A} + \mathfrak{B} = \mathfrak{A} + q_0$ is the set of all points $p + q_0$, where p is any point in $\mathfrak{A}$. Notice that $\mathfrak{A} + q_0$ is obtained by translating A by the vector $\overrightarrow{Oq_0}$.

We can also define the operation of multiplication by a real number s.

DEFINITION. *The* product figure, $s\mathfrak{A}$, *is the set of all points sp where p belongs to $\mathfrak{A}$.*

Figure $s\mathfrak{A}$ is obtained from $\mathfrak{A}$ by a similarity transformation with coefficient s and center at the origin.

In general, if we are given figures $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_k$ and k real numbers $s_1, s_2, \ldots, s_k$, then

$$s_1\mathfrak{A}_1 + s_2\mathfrak{A}_2 + \cdots + s_k\mathfrak{A}_k$$

is the set of all points

$$s_1p_1 + s_2p_2 + \cdots + s_kp_k$$

where p_i belongs to $\mathfrak{A}_i$.

We shall study only the case in which we are given two figures $\mathfrak{Q}$ and $\mathfrak{Q}_1$ and two positive numbers s and s_1 with sum $s + s_1 = 1$. Then we write

$$\mathfrak{Q}_s = s\mathfrak{Q} + s_1\mathfrak{Q}_1 = s\mathfrak{Q} + (1 - s)\mathfrak{Q}_1.$$

Thus, $\mathfrak{Q}_s$ is the set of all points of the form $q_s = sq + s_1q_1$, where q belongs to $\mathfrak{Q}$ and q_1 belongs to $\mathfrak{Q}_1$. Geometrically, $\mathfrak{Q}_s$ is the locus of all points q_s dividing the segments qq_1 in the ratio s_1/s.

DEFINITION. *The operation of forming $\mathfrak{Q}_s$ as shown above is called* mixing $\mathfrak{Q}$ and $\mathfrak{Q}_1$.

If $s = s_1 = \frac{1}{2}$, then $\mathfrak{D}_s = \mathfrak{D}_{\frac{1}{2}}$ is the locus of the midpoints of segments connecting all points of $\mathfrak{D}$ with all points of $\mathfrak{D}_1$.

EXAMPLE 1. Let l and l_1 be parallel lines (Fig. 108) given by the normal equations

$$x \cos \alpha + y \sin \alpha = h, \tag{1}$$

$$x \cos \alpha + y \sin \alpha = h_1. \tag{2}$$

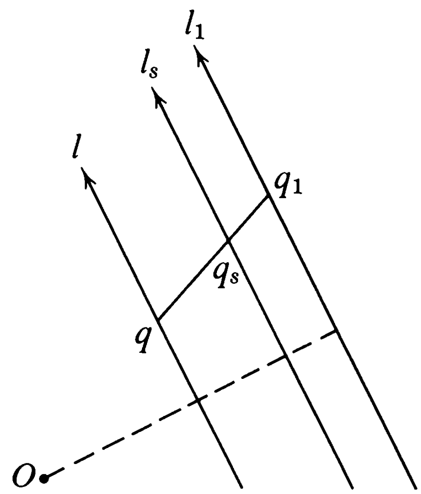


Fig. 108

Then

$$l_s = sl + s_1 l_1 \quad (s + s_1 = 1;\ s, s_1 \geq 0)$$

is a line parallel to l and l_1 given by the equation

$$x \cos \alpha + y \sin \alpha = h_s, \tag{3}$$

where

$$h_s = sh + s_1 h_1.$$

Proof. Let q_s be an arbitrary point on line l_s. Then

$$q_s = sq + s_1 q_1$$

where q is on l and q_1 is on l_1. Let (x, y), (x_1, y_1), and (x_s, y_s) be the coordinates of points q, q_1, and q_s respectively. Then from (3) of the preceding section, we have

$$x_s = sx + s_1 x_1,$$

$$y_s = sy + s_1 y_1.$$

The coordinates (x, y) of point q satisfy equation (1); thus,

$$x \cos \alpha + y \sin \alpha = h.$$

Analogously,

$$x_1 \cos \alpha + y_1 \sin \alpha = h_1.$$

Consequently,

$$\begin{aligned} x_s \cos \alpha + y_s \sin \alpha &= (sx + s_1x_1) \cos \alpha + (sy + s_1y_1) \sin \alpha \\ &= s(x \cos \alpha + y \sin \alpha) \\ &\qquad\qquad + s_1(x_1 \cos \alpha + y_1 \sin \alpha) \\ &= sh + s_1h_1 = h_s; \end{aligned}$$

that is, the coordinates of point q_s satisfy equation (3).

Hence, if we connect all points q on the line l with all points q_1 on the line l_1 by segments qq_1, and on each of these segments, pick out the point q_s which divides the segment in the ratio s_1/s, then the set of all points q_s is the line l_s (Fig. 108).

Remark. Now suppose that the plane figures $\mathfrak{Q}$ and $\mathfrak{Q}_1$ do not lie in one plane, but in two parallel planes R and R_1. Let every point q of figure $\mathfrak{Q}$ be connected with every point q_1 of $\mathfrak{Q}_1$ by line segments qq_1. The set of all these segments forms some solid T, contained between planes R and R_1. Now construct plane R_s parallel to R and R_1, and dividing the distance between them in the ratio s_1/s. The intersection of this plane and T gives us $\mathfrak{Q}_s = s\mathfrak{Q} + s_1\mathfrak{Q}_1$. In fact, the point of intersection of the plane R_s and each segment qq_1 divides this segment in the ratio s_1/s. Obviously, the entire plane R_s may be defined as a linear combination of planes R and R_1:

$$R_s = sR + s_1R_1.$$

Example 2. Now we shall consider the lines l and l_1 to be not only parallel, but also to have the same direction (Fig. 108). Each of the lines l, l_1, and l_s divides the plane into two half-planes. l divides the plane into half-plane A, lying on the right, and half-plane B, lying on the left of l. Analogously, l_1 divides the plane into half-planes A_1 and B_1, lying on the right and left of l_1, and l_s divides the plane into half-planes A_s and B_s, lying on the right and left of l_s. We can show that

$$A_s = sA + s_1A_1, \tag{4}$$

$$B_s = sB + s_1B_1. \tag{5}$$

Proof. Half-plane A is the set of all points whose coordinates (x, y) satisfy the inequality

$$x \cos \alpha + y \sin \alpha \geq h. \tag{6}$$

Half-planes A_1 and A_s are the sets of points (x_1, y_1) and (x_s, y_s) whose coordinates satisfy the inequalities

$$x_1 \cos \alpha + y_1 \sin \alpha \geq h_1, \tag{7}$$

$$x_s \cos \alpha + y_s \sin \alpha \geq h_s. \tag{8}$$

Linear combinations $p_s = sp + s_1p_1$ for which the coordinates of points $p:(x, y)$ and $p_1:(x_1, y_1)$ satisfy inequalities (6) and (7) yield points $p_s:(x_s, y_s)$, whose coordinates satisfy inequality (8).

If $s = s_1 = \frac{1}{2}$, then line $l_{\frac{1}{2}}$ is the locus of the midpoints of all line segments connecting points of lines l and l_1. Half-plane $A_{\frac{1}{2}}$ lying on the right of $l_{\frac{1}{2}}$ is composed of the midpoints of all segments connecting points of half-planes A and A_1. This is apparent geometrically.

Example 3. Consider two parallel segments AB and A_1B_1 with the same direction lying on lines l and l_1 (Fig. 109). Denote these segments by k and k_1, and form the figure

$$k_s = sk + s_1k_1 \quad (s + s_1 = 1;\ s, s_1 \geq 0). \tag{9}$$

Figure k_s lies on the line $l_s = sl + s_1l_1$. As shown in Example 1, l_s is parallel to l and l_1. The equations of lines l, l_1, and l_s are

$$\left.\begin{aligned} x \cos \alpha + y \sin \alpha &= h, \\ x \cos \alpha + y \sin \alpha &= h_1, \\ x \cos \alpha + y \sin \alpha &= h_s, \end{aligned}\right\} \quad \text{where } h_s = sh + s_1h_1. \tag{10}$$

Figure k_s is the locus of points q_s dividing segments qq_1 in the ratio s_1/s, where q is on k and q_1 is on k_1. As shown in Figure 109, the points q_s form segment A_sB_s of line l_s, where A_s is the point dividing segment AA_1 in the ratio s_1/s and B_s divides BB_1 in the same ratio. If a, a_1, and a_s denote the lengths of segments k, k_1, and k_s, then

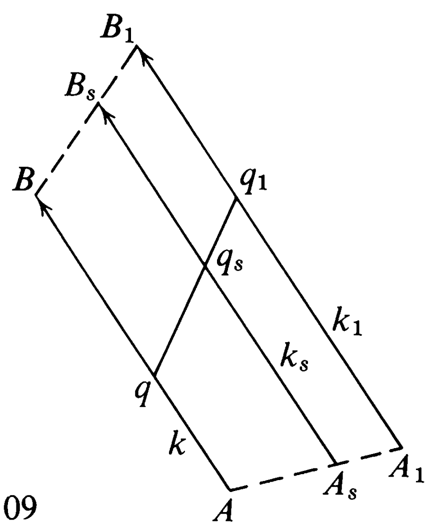


Fig. 109

$$a_s = sa + s_1a_1. \tag{11}$$

Equations (10) and (11) show that the relationship between the lengths a, a_1, and a_s of segments k, k_1, and k_s and between the distances h, h_1, and h_s from the origin to the lines on which these segments lie is the same as the relationship (9) between the segments themselves.

We can now prove the following fundamental theorem.

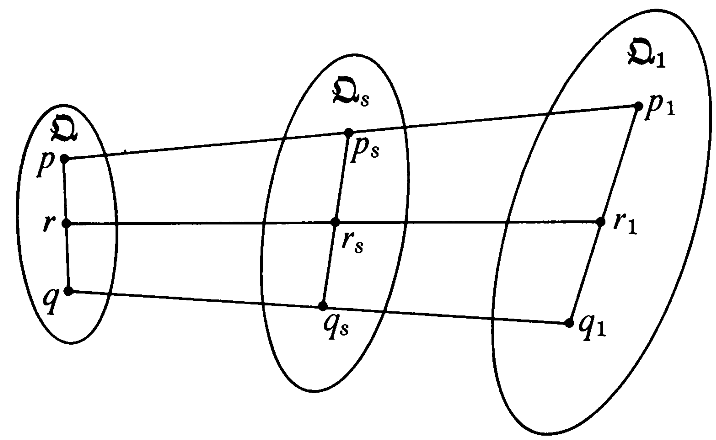
THEOREM 1. *If* $\mathfrak{Q}$ *and* $\mathfrak{Q}_1$ *are convex figures, then*

$$\mathfrak{Q}_s = s\mathfrak{Q} + s_1\mathfrak{Q}_1 \quad (s + s_1 = 1;\ s, s_1 \geq 0)$$

is also a convex figure.

Proof. Suppose that p_s and q_s are two points of $\mathfrak{Q}_s$ (Fig. 110).

Fig. 110



Then they may be represented in the form

$$p_s = sp + (1 - s)p_1, \quad q_s = sq + (1 - s)q_1, \tag{12}$$

where p and q are points of figure $\mathfrak{Q}$, and p_1 and q_1 are points of figure $\mathfrak{Q}_1$.

Segment p_sq_s is the set of all points of the form

$$r_s = tq_s + t_1p_s \quad (t + t_1 = 1;\ t, t_1 \geq 0). \tag{13}$$

From equations (12) and (13) we obtain

$$\begin{aligned} r_s &= t(sq + s_1q_1) + t_1(sp + s_1p_1) \\ &= s(tq + t_1p) + s_1(tq_1 + t_1p_1) = sr + s_1r_1 \end{aligned}$$

where r and r_1 are given by

$$\begin{aligned} r &= tq + t_1p, \\ r_1 &= tq_1 + t_1p_1. \end{aligned}$$

Thus r is on segment pq. Since $\mathfrak{Q}$ is convex, it contains this segment, and must also contain r. Similarly, r_1 belongs to $\mathfrak{Q}_1$.

We have shown that r_s is of the form $sr + s_1r_1$ where r belongs to $\mathfrak{Q}$ and r_1 to $\mathfrak{Q}_1$. But $\mathfrak{Q}_s$ is by definition the set of all points having this form. Hence, r_s belongs to $\mathfrak{Q}_s$. Since point r_s is an arbitrary point of the segment p_sq_s, then the entire segment p_sq_s belongs to $\mathfrak{Q}_s$. Therefore $\mathfrak{Q}_s$ is convex and the theorem is proved.

THEOREM 2. *Let $\mathfrak{Q}$ and $\mathfrak{Q}_1$ be plane convex figures and let l and l_1 be parallel supporting lines for them having the same direction. Form the figure $\mathfrak{Q}_s = s\mathfrak{Q} + s_1\mathfrak{Q}_1$ $(s + s_1 = 1;\ s, s_1 \geq 0)$ and the line $l_s = sl + s_1l_1$. Then l_s is a supporting line for $\mathfrak{Q}_s$ parallel to l and l_1 and having the same direction.*

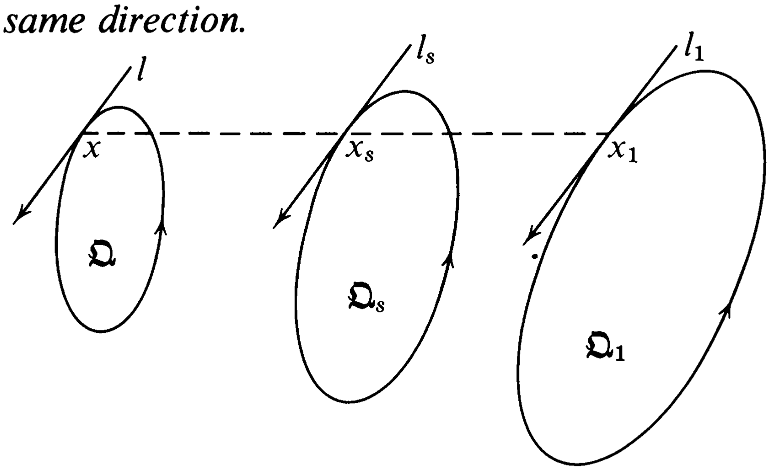


Fig. 111

Proof. Suppose that $\mathfrak{Q}$ lies to the left of l and $\mathfrak{Q}_1$ lies to the left of l_1 (Fig. 111). From Example 1, l_s is a line parallel to l and l_1 and having the same direction. From Example 2, we conclude that $\mathfrak{Q}_s$ lies to the left of l_s.

Line l and figure $\mathfrak{Q}$ must have some point x in common, because l is a supporting line for $\mathfrak{Q}$. For the same reason, l_1 and $\mathfrak{Q}_1$ have a common point x_1. Now consider the point $x_s = sx + s_1x_1$. It belongs to both l_s and $\mathfrak{Q}_s$. Consequently, l_s is a supporting line for $\mathfrak{Q}_s$ and the theorem is proved.

THEOREM 3. *If figures $\mathfrak{Q}$ and $\mathfrak{Q}_1$ are translated by vectors $\overrightarrow{Oa}$ and $\overrightarrow{Oa_1}$ respectively, then $\mathfrak{Q}_s = s\mathfrak{Q} + s_1\mathfrak{Q}_1$ is translated by the vector $\overrightarrow{Oa_s}$, where $a_s = sa + s_1a_1$.*

Proof. $\mathfrak{Q}$ is transformed into $\mathfrak{Q} + a$, $\mathfrak{Q}_1$ into $\mathfrak{Q}_1 + a_1$. Then $\mathfrak{Q}_s = s\mathfrak{Q} + s_1\mathfrak{Q}_1$ is transformed into

$$s(\mathfrak{Q} + a) + s_1(\mathfrak{Q}_1 + a_1) = (s\mathfrak{Q} + s_1\mathfrak{Q}_1) + (sa + s_1a_1) = \mathfrak{Q}_s + a_s,$$

and the theorem is proved.

27. LINEAR SYSTEMS OF CONVEX POLYGONS; AREAS AND "MIXED AREAS"

Consider two polygons Q and Q_1 in which corresponding sides are parallel and have the same directions (for example, polygons $A_1A_2 \ldots A_iA_{i+1} \ldots A_n$ and $A_1'A_2' \ldots A_i'A'_{i+1} \ldots A_n'$ in Figure 112).

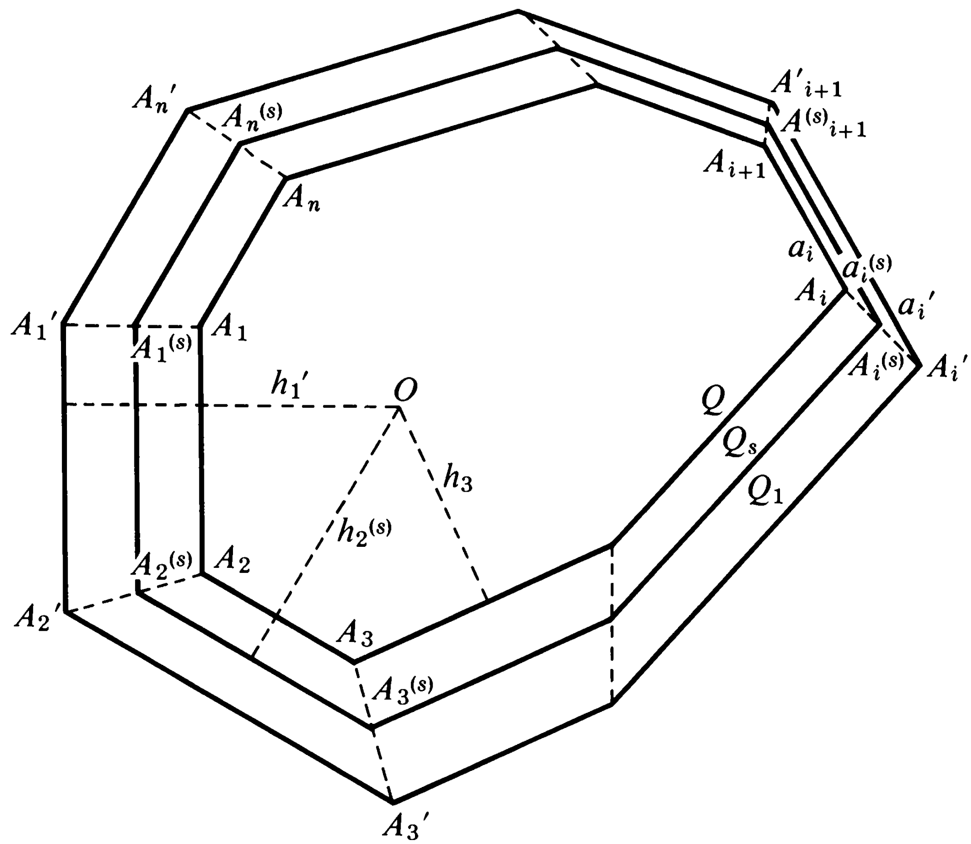


Fig. 112

We shall construct polygon $Q_s = sQ + s_1Q_1$ where $s + s_1 = 1$ and $s, s_1 \geq 0$ as usual. Let a_i be the length of side A_iA_{i+1} of the first polygon and a_i' that of side $A_i'A'_{i+1}$ of the second. Form the linear combinations

$$a_i^{(s)} = sa_i + s_1a_i'$$

for $i = 1, 2, \ldots, n$. By Example 3, section 26, $a_i^{(s)}$ is the length of segment $A_i^{(s)}A_{i+1}^{(s)}$ where

$$\begin{aligned} A_i^{(s)} &= sA_i + s_1A_i', \\ A_{i+1}^{(s)} &= sA_{i+1} + s_1A'_{i+1}. \end{aligned}$$

Let l and l' be the supporting lines for Q and Q_1 which contain segments a_i and a_i' respectively. Then $l_s = sl + s_1 l'$ is a supporting line for Q_s. Since segment $A_i^{(s)}A_{i+1}^{(s)}$ belongs to both Q_s and its supporting line l_s, then it is part of the boundary of Q_s. *The boundary of Q_s*

consists of the segments $A_i^{(s)}A_{i+1}^{(s)}$, *parallel to and having the same direction as the corresponding sides of* Q *and* Q_1. Thus Q_s is a polygon with vertices $A_i^{(s)}$ ($i = 1, 2, \ldots, n$) and sides $A_i^{(s)}A_{i+1}^{(s)}$ with lengths $a_i^{(s)}$.

If h_i, h_i', and $h_i^{(s)}$ are the perpendicular distances from the origin O to sides a_i, a_i', and $a_i^{(s)}$ of polygons Q, Q_1, and Q_s, then

$$h_i^{(s)} = sh_i + s_1h_i'$$

(see Example 3, section 26).

We drew Figure 112 with the origin inside both Q and Q_1. If it were not, we could make it so by translating Q and Q_1. As shown in Theorem 3, section 26, Q_s would also be translated.

Polygon Q_s can be constructed as follows: First, connect vertices A_1 and A_1' of polygons Q and Q_1 with segment A_1A_1' and divide this segment in the ratio s_1/s, to obtain $A_1^{(s)}$. From $A_1^{(s)}$ draw a line parallel to A_1A_2. When sufficiently extended, it will meet segment A_2A_2' at the point $A_2^{(s)}$, a vertex of Q_s. From $A_2^{(s)}$ draw a line parallel to A_2A_3 and extend it to meet segment A_3A_3' in $A_3^{(s)}$. Continue in this manner to construct the remaining vertices and sides of Q_s; the last to be constructed will be the side $A_n^{(s)}A_1^{(s)}$.

Now suppose that polygons Q and Q_1 are not composed of pairs of corresponding parallel sides. For example, in Figure 113, Q is the

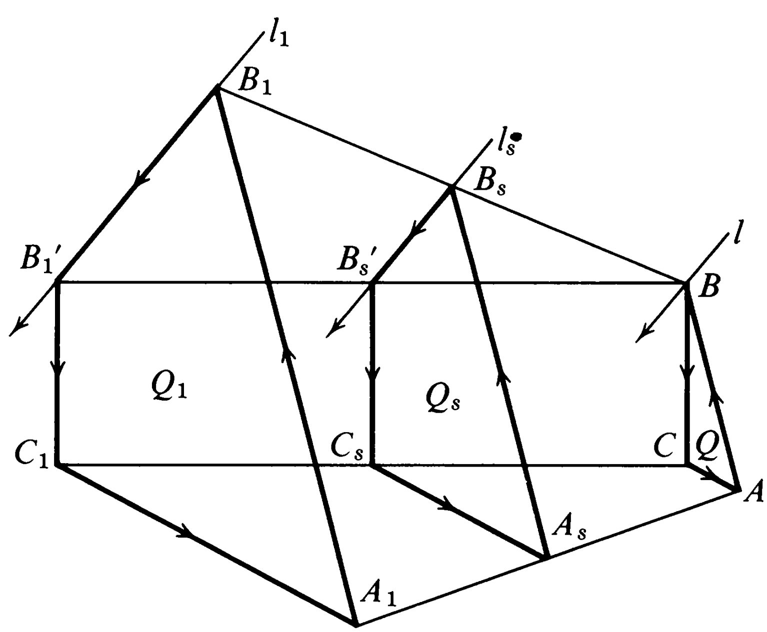


Fig. 113

triangle ABC and Q_1 is the quadrilateral $A_1B_1B_1'C_1$; sides AB, BC, and CA of Q are parallel to and have the same directions as sides A_1B_1, $B_1'C_1$, and C_1A_1 of Q_1, but Q has no side parallel to and having the same direction as B_1B_1'.

Side B_1B_1' is part of a supporting line l_1 for polygon Q_1. Polygon Q has a supporting line l parallel to l_1 and having the same direction. Line l passes through only one point of Q, that is, the vertex B. We then say that polygon Q has a *degenerate* side BB' parallel to B_1B_1' and having the same direction. Side BB' is called degenerate because its end points coincide. The addition of this degenerate side reduces the present case to the preceding case, in which polygons Q and Q_1 consisted of pairs of parallel and similarly directed sides.

Then $Q_s = sQ + s_1Q_1$ is a polygon $A_sB_sB_s'C_s$ in which sides A_sB_s, B_sB_s', $B_s'C_s$, and C_sA_s are parallel to and have the same directions as the corresponding sides of Q and Q_1. The vertices A_s, B_s, B_s', C_s divide segments AA_1, BB_1, $B'B_1'$, CC_1 in the ratio s_1/s.

We shall now apply this method to the general case. Suppose that we are given two polygons Q and Q_1. Suppose also that Q_1 does not have a side parallel to and with the same direction as side A_iA_{i+1} of Q. Draw the supporting line l_i' for Q_1 parallel to A_iA_{i+1} and having the same direction. This line passes through only one vertex A' of polygon Q_1. Now we say that polygon Q_1 has a degenerate side $A_i'A'_{i+1}$ parallel to A_iA_{i+1} and having the same direction (both end points of $A_i'A'_{i+1}$ coincide with point A'). We do the same for Q if it fails to have a side parallel to and with the same direction as some side of Q_1.

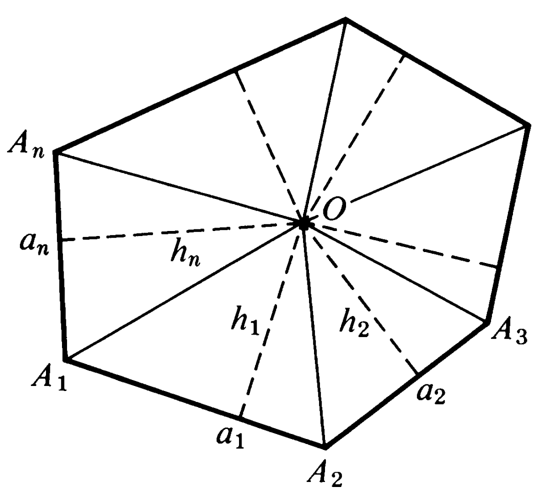
By adding the necessary degenerate sides as shown above, we may say:

Any two polygons are composed of pairs of parallel corresponding sides with the same directions.

The analogous situation for polyhedra is used in Chapter 5.

We shall now compute the area of a convex polygon Q with vertices $A_1, A_2, \ldots, A_n$ (Fig. 114). We translate Q so that the origin

Fig. 114



O lies inside Q. The ith side A_iA_{i+1} has an equation of the form

$$x \cos \alpha + y \sin \alpha = h_i$$

where h_i is the distance from the origin O to the side A_iA_{i+1}.

If the length of A_iA_{i+1} is a_i, then the area of triangle OA_iA_{i+1} is half the product of the length a_i of the base A_iA_{i+1} and the altitude h_i.

$$\text{Area of } OA_iA_{i+1} = \frac{1}{2} a_ih_i.$$

Since the entire polygon Q may be divided into n such triangles, then its area, $J(Q)$, is given by

$$J(Q) = \frac{1}{2} \sum_{i=1}^{n} a_ih_i. \tag{1}$$

Now consider two polygons Q and Q_1 (Fig. 115) with parallel and similarly directed corresponding sides. (As we have just seen, any two polygons may be so considered upon the addition of certain degenerate sides. This does not affect the computation of the area, because the length of a degenerate side is zero.) Denote the vertices of Q by $A_1, A_2, \ldots, A_n$, and the corresponding vertices of Q_1 by $A_1', A_2', \ldots, A_n'$. Let a_i and a_i' be the lengths of sides A_iA_{i+1} and

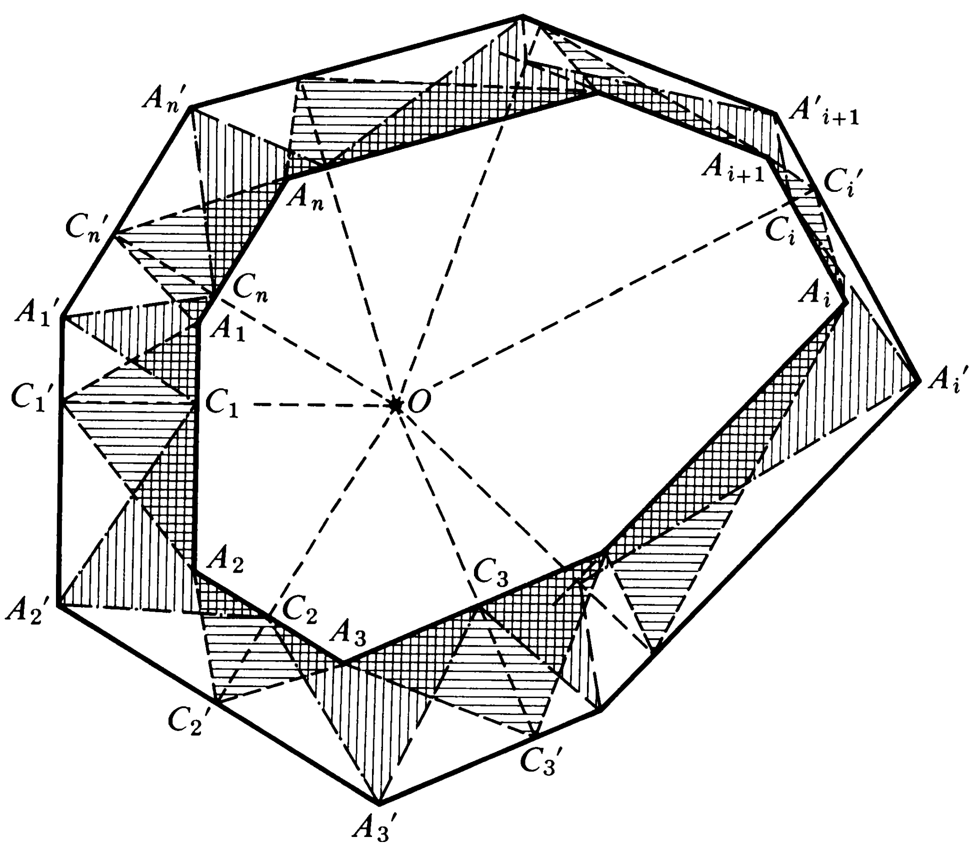


Fig. 115

$A_i'A'_{i+1}$. (These sides are parallel and similarly directed.) Sides A_iA_{i+1} and $A_i'A'_{i+1}$ have equations of the form

$$x \cos \alpha + y \sin \alpha = h_i,$$
$$x \cos \alpha + y \sin \alpha = h_i'.$$

By formula (1) the areas $J(Q)$ and $J(Q_1)$ are given by

$$J(Q) = \frac{1}{2} \sum_{i=1}^{n} a_i h_i, \tag{2}$$

$$J(Q_1) = \frac{1}{2} \sum_{i=1}^{n} a_i' h_i'. \tag{2'}$$

We shall now compute the area of the polygon

$$Q_s = sQ + s_1 Q_1.$$

By the previous results, polygon Q_s has vertices

$$A_i^{(s)} = sA_i + s_1 A_i',$$

and the length of side $A_i^{(s)}A_{i+1}^{(s)}$ is

$$a_i^{(s)} = sa_i + s_1 a_i'.$$

The equation of this side is

$$x \cos \alpha + y \sin \alpha = h_i^{(s)},$$

where

$$h_i^{(s)} = sh_i + s_1 h_i'.$$

From formula (1) the area of Q_s is

$$J(Q_s) = \frac{1}{2} \sum_{i=1}^{n} a_i^{(s)} h_i^{(s)} = \frac{1}{2} \sum_{i=1}^{n} (sa_i + s_1 a_i')(sh_i + s_1 h_i'). \tag{3}$$

Eliminating the parentheses under the summation sign in equation (3) and collecting the terms with s^2, ss_1, and s_1^2, we obtain

$$J(Q_s) = s^2\left(\frac{1}{2} \sum_{i=1}^{n} a_i h_i\right) + ss_1\left(\frac{1}{2} \sum_{i=1}^{n} a_i h_i' + \frac{1}{2} \sum_{i=1}^{n} a_i' h_i\right) + s_1^2\left(\frac{1}{2} \sum_{i=1}^{n} a_i' h_i'\right). \tag{4}$$

By formulas (2) and (2') the coefficients of s^2 and s_1^2 in the right side of formula (4) are equal to $J(Q)$ and $J(Q_1)$. We shall study in detail the coefficient of ss_1.

We first prove the equality of the two sums appearing in the coefficient of ss_1:

$$\frac{1}{2}\sum_{i=1}^{n} a_i h_i' = \frac{1}{2}\sum_{i=1}^{n} a_i' h_i.$$

Drop perpendiculars from point O to the sides of polygons Q and Q_1. (The perpendiculars to a pair of parallel sides with the same direction will coincide.) For simplicity, we may assume that O lies inside both Q and Q_1 (Fig. 115), because it may always be made to do so by suitable translations of these polygons. Let $C_1, C_2, \ldots, C_n$ denote the feet of the perpendiculars dropped from O to sides $A_1A_2, A_2A_3, \ldots, A_nA_1$ of polygon Q. Similarly, let $C_1', C_2', \ldots, C_n'$ denote the feet of the perpendiculars to the corresponding sides of polygon Q_1. Then we have

$$\begin{aligned} OC_1 &= h_1, & OC_2 &= h_2, & \ldots, && OC_n &= h_n; \\ OC_1' &= h_1', & OC_2' &= h_2', & \ldots, && OC_n' &= h_n'. \end{aligned}$$

By connecting each point C_i' with the two neighboring vertices A_i, A_{i+1} of polygon Q, we obtain the polygon $A_1C_1'A_2C_2' \ldots A_nC_n'A_1$ bounded by the dashed line in Figure 115. Its area is

$$\frac{1}{2}\sum_{i=1}^{n} a_i h_i'.$$

To prove this, observe that the area of the polygon is the sum of the areas of quadrilaterals $OA_1C_1'A_2, OA_2C_2'A_3, \ldots, OA_nC_n'A_1$. The area of quadrilateral $OA_1C_1'A_2$ with perpendicular diagonals OC_1' and A_1A_2 is half the product of the lengths of these diagonals, or $\frac{1}{2}a_1h_1'$. Similarly, the area of the next quadrilateral is $\frac{1}{2}a_2h_2'$, and so forth. Therefore the total area of our polygon is as stated.

We now construct the polygon $A_1'C_1A_2'C_2 \ldots A_n'C_nA_1'$, whose boundary is represented in Figure 115 by the line consisting of dots and dashes. As above, we can prove that the area of this polygon is

$$\frac{1}{2}\sum_{i=1}^{n} a_i' h_i.$$

The polygon bounded by the ordinary dashed line is obtained from polygon Q by adding to it the $2n$ triangles (shaded by horizontal lines)

$$A_1C_1'C_1,\ C_1C_1'A_2,\ A_2C_2'C_2,\ C_2C_2'A_3,\ \ldots,\ A_nC_n'C_n,\ C_nC_n'A_1.$$

The polygon bounded by the line of dots and dashes is obtained from polygon Q by adding to it the $2n$ triangles (shaded by vertical lines)

$$A_1A_1'C_1,\ C_1A_2'A_2,\ A_2A_2'C_2,\ C_2A_3'A_3,\ \ldots,\ A_nA_n'C_n,\ C_nA_1'A_1.$$

Notice that the areas of the triangles in the above sequences are pairwise equal. For example, triangles $A_1A_1'C_1$ and $A_1C_1'C_1$ have common base A_1C_1, and the vertices A_1' and C_1' opposite it lie on a line parallel to the base, giving the triangles equal altitudes. Therefore the areas of the two triangles are equal. Similarly the remaining pairs of triangles are equal: $C_1A_2'A_2$ and $C_1C_1'A_2$, $A_2A_2'C_2$ and $A_2C_2'C_2$, etc.

The areas of the figure bounded by the dashed line and the figure bounded by the line of dots and dashes are equal, because these figures consist of pairwise equal parts. It follows that

$$\frac{1}{2}\sum_{i=1}^{n} a_ih_i' = \frac{1}{2}\sum_{i=1}^{n} a_i'h_i.$$

Definition. *The expression* $\frac{1}{2}\sum_{i=1}^{n} a_ih_i'$ *or* $\frac{1}{2}\sum_{i=1}^{n} a_i'h_i$ *is called the* mixed area of polygons Q and Q_1. *It is denoted by* $J(Q, Q_1)$.

Thus, formula (4) takes the form

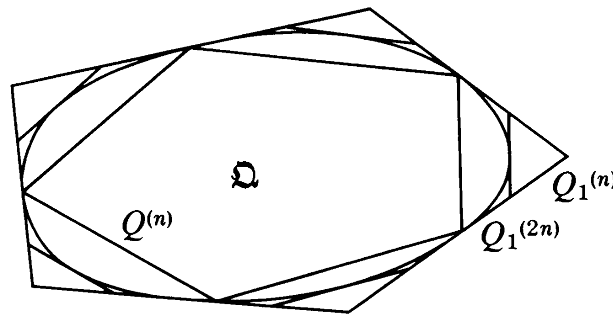
$$J(Q_s) = s^2J(Q) + 2ss_1J(Q, Q_1) + s_1^2J(Q_1). \tag{4'}$$

Notice that the mixed area of polygons Q and Q_1 may be obtained from formula (1) for the area of a polygon by using the lengths of the sides of one polygon and the lengths of the perpendiculars dropped from point O to the sides of the other polygon. If $Q = Q_1$, then the mixed area $J(Q, Q_1)$ of Q and Q_1 coincides with the ordinary area of Q.

In courses in elementary geometry the area (or the circumference) of a circle is defined as the limit of the areas (or perimeters) of inscribed and circumscribed polygons. The area of any plane convex figure $\mathfrak{Q}$ and the length of its boundary q are defined analogously (cf. sections 6 and 12). For each integer $n \geq 3$, we construct an in-

scribed polygon $Q^{(n)}$ with n sides (Fig. 116). As n, the number of sides, increases without bound and the lengths of the sides approach zero, the polygons $Q^{(n)}$ approach the shape of figure $\mathfrak{Q}$. The areas $J(Q^{(n)})$ approach a limit which we denote by $J(\mathfrak{Q})$ and define to be the area of figure $\mathfrak{Q}$. We can also construct circumscribed polygons $Q_1{}^{(n)}$ about $\mathfrak{Q}$. As n increases without bound and the $Q_1{}^{(n)}$ converge to $\mathfrak{Q}$, their areas $J(Q_1{}^{(n)})$ approach the same limit $J(\mathfrak{Q})$.

Fig. 116



Let $q^{(n)}$ and $q_1{}^{(n)}$ denote the boundaries of $Q^{(n)}$ and $Q_1{}^{(n)}$. As $n \to \infty$, the lengths $l(q^{(n)})$ and $l(q_1{}^{(n)})$ of the perimeters $q^{(n)}$ and $q_1{}^{(n)}$ approach a common limit $l(q)$, which we define to be the length of q. Thus

$$J(\mathfrak{Q}) = \lim_{n\to\infty} J(Q^{(n)}) = \lim_{n\to\infty} J(Q_1{}^{(n)}),$$

$$l(q) = \lim_{n\to\infty} l(q^{(n)}) = \lim_{n\to\infty} l(q_1{}^{(n)}).$$

We shall not give the details of the rigorous basis necessary for these definitions. This would be done in essentially the same way as the area of a circle and its circumference are presented in elementary courses in geometry.

Now we shall consider the three plane convex figures

$$\mathfrak{Q},\quad \mathfrak{Q}_1,\quad \mathfrak{Q}_s = s\mathfrak{Q} + s_1\mathfrak{Q}_1 \quad (s + s_1 = 1;\ s, s_1 \geq 0).$$

We shall construct a sequence of inscribed (or circumscribed) polygons $Q^{(n)}$ converging to $\mathfrak{Q}$, and another sequence $Q_1{}^{(n)}$ converging to $\mathfrak{Q}_1$. Then the polygons $Q_s{}^{(n)} = sQ^{(n)} + s_1Q_1{}^{(n)}$ will converge to Q_s. Thus

$$J(\mathfrak{Q}) = \lim_{n\to\infty} J(Q^{(n)}),$$

$$J(\mathfrak{Q}_1) = \lim_{n\to\infty} J(Q_1{}^{(n)}),$$

$$J(\mathfrak{Q}_s) = \lim_{n\to\infty} J(Q_s{}^{(n)}).$$

On the other hand, from formula (4′), which is proved for all polygons, we obtain

$$J(Q_s^{(n)}) = s^2J(Q^{(n)}) + 2ss_1J(Q^{(n)}, Q_1^{(n)}) + s_1^2J(Q_1^{(n)}). \qquad (5)$$

As n increases without bound, the coefficient of s^2 in formula (5) approaches $J(\mathfrak{Q})$, the coefficient of s_1^2 approaches $J(\mathfrak{Q}_1)$, and the term on the left appraches $J(\mathfrak{Q}_s)$. The remaining term, containing $2ss_1$, must also approach some limit, that is, the mixed area $J(Q^{(n)}, Q_1^{(n)})$ of $Q^{(n)}$ and $Q_1^{(n)}$ approaches some limit as $n \to \infty$. It is natural to define the *mixed area of convex figures* $\mathfrak{Q}$ *and* $\mathfrak{Q}_1$ to be this limit and write

$$J(\mathfrak{Q}, \mathfrak{Q}_1) = \lim_{n\to\infty} J(Q^{(n)}, Q_1^{(n)}).$$

Taking the limits in formula (5), we obtain

$$J(\mathfrak{Q}_s) = s^2J(\mathfrak{Q}) + 2ss_1J(\mathfrak{Q}, \mathfrak{Q}_1) + s_1^2J(\mathfrak{Q}_1). \qquad (5')$$

This looks exactly like formula (4′), which was proved for convex polygons only. Now we have shown that it is also valid for arbitrary plane convex figures.

The concept of mixed areas turns out to be extremely fruitful. We shall show in the next section that several geometric quantities associated with a convex figure $\mathfrak{Q}$ may be defined as the mixed area of $\mathfrak{Q}$ and some other figure.

28. APPLICATIONS

EXAMPLE 1. *Length of projection as a mixed area.* Consider an arbitrary polygon Q and a line segment $l = AB$ of length 1 (Fig. 117). Polygon Q has two sides with opposite directions parallel to l. (As above, these sides may have length zero.) Denote these sides by b_i and b_j. Segment l may be considered as a degenerate polygon L with two sides b_i' and b_j' of length 1, parallel to b_i and b_j and having the same directions. (One side is directed from A to B, and the other from B to A.) Any additional sides of this degenerate polygon have length zero. Polygon L has only two vertices, A and B.

If b_k' is a side of polygon L, then we denote its length by a_k'. Now we have

$$a_i' = 1, \quad a_j' = 1, \quad \text{and} \quad a_k' = 0,$$

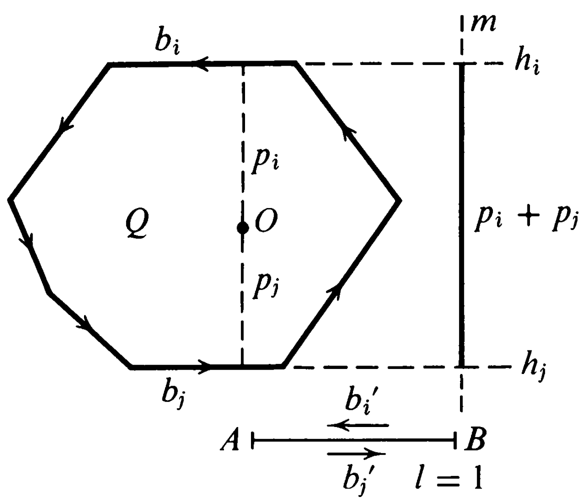


Fig. 117

where k is different from i and j. Now consider $J(Q, L)$. We know that

$$J(Q, L) = \frac{1}{2} \sum_{k=1}^{n} a_k' p_k,$$

where p_k denotes the length of the perpendicular dropped from O to side b_k of polygon Q. Since all $a_k' = 0$ except a_i' and a_j', which equal 1, we have

$$J(Q, L) = \frac{1}{2}(p_i + p_j).$$

Now draw supporting lines h_i and h_j for polygon Q parallel to segment l. Sides b_i and b_j lie on these lines; p_i is the distance from O to h_i, p_j is the distance from O to h_j, and $p_i + p_j$ is the distance between h_i and h_j. If m is a straight line perpendicular to l, then $p_i + p_j$ is the length of the projection of figure Q on m. Thus, we have proved the following proposition for the case that Q is a polygon. In passing to limits, Q may be replaced by an arbitrary convex figure, and we can prove the validity of the proposition for all convex figures.

THEOREM 4. *If l is a line segment with length* 1 *and m is a line perpendicular to l, then the mixed area of convex figure $\mathfrak{Q}$ and l is numerically equal to half the length of the projection of $\mathfrak{Q}$ on m.*

EXAMPLE 2. *Perimeter as a mixed area.* Now consider an arbitrary convex polygon Q and let Q_1 be a convex polygon circumscribed about a circle with radius 1 so that the sides of Q_1 are parallel to the corresponding sides of Q and have the same directions (Fig. 118).

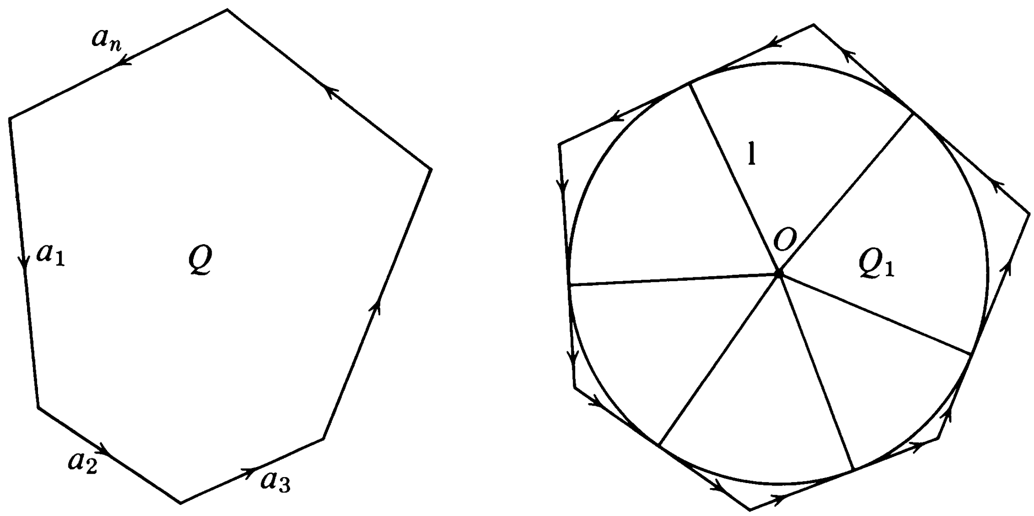


Fig. 118

Given polygon Q, it is always possible to construct a corresponding polygon Q_1 satisfying the above conditions. First, draw a circle with center at O and radius 1. Then draw tangent lines to this circle parallel to the sides of Q and having the same direction. These tangents form the boundary of the desired polygon Q_1.

The mixed area $J(Q, Q_1)$ is given by

$$J(Q, Q_1) = \frac{1}{2} \sum_{i=1}^{n} a_i p_i',$$

where n is the number of sides of Q, a_i is the length of the ith side, and p_i' is the length of the perpendicular dropped from O to the corresponding side of Q_1. Since the sides of Q_1 are tangent to a circle with radius 1 and center at O, then all $p_i' = 1$.

Therefore

$$J(Q, Q_1) = \frac{1}{2} \sum_{i=1}^{n} a_i.$$

The right side of this equation is half the sum of the lengths of the sides of polygon Q, that is, half the perimeter of Q. Thus, we have:

THEOREM 5. *The mixed area of convex polygons Q and Q_1, where Q_1 is circumscribed about a circle with radius* 1 *and has sides parallel to the sides of Q and with the same directions, is numerically equal to half the perimeter of Q.*

Using Theorem 5, we can prove the following:

THEOREM 6. *The mixed area of a plane convex figure $\mathfrak{Q}$ and a circle R with radius* 1 *is numerically equal to half the length d of the boundary of $\mathfrak{Q}$;*

$$J(\mathfrak{Q}, R) = \frac{1}{2} d.$$

Proof. Here $\mathfrak{Q}$ is an arbitrary plane convex figure, R a circle with radius 1, and d the length of the boundary of $\mathfrak{Q}$. Around $\mathfrak{Q}$ and R, construct sequences $Q^{(n)}$ and $R^{(n)}$ of circumscribed polygons with corresponding sides parallel and similarly directed, so that as n increases without bound, polygons $Q^{(n)}$ converge to $\mathfrak{Q}$, and $R^{(n)}$ to R.

By Theorem 5,

$$J(Q^{(n)}, R^{(n)}) = \frac{1}{2} d^{(n)},$$

where $d^{(n)}$ is the length of the boundary (perimeter) of $Q^{(n)}$. As n increases without bound, $d^{(n)}$ approaches d and $J(Q^{(n)}, R^{(n)})$ approaches $J(\mathfrak{Q}, R)$, and we have the desired result.

29. SCHWARZ INEQUALITY; OTHER INEQUALITIES

In the next section, we shall prove an inequality relating the areas $J(\mathfrak{Q})$, $J(\mathfrak{Q}_1)$, and $J(\mathfrak{Q}, \mathfrak{Q}_1)$. First, we must prove certain other inequalities, which, in turn, are connected with some inequalities concerning the coefficients of a quadratic equation.

If we are given the quadratic equation

$$at^2 + 2bt + c = 0, \tag{1}$$

then its roots have the form

$$\frac{1}{a}(-b \pm \sqrt{b^2 - ac}).$$

(i) If $b^2 - ac < 0$, the roots of equation (1) are complex. There is no real root; that is, the expression

$$at^2 + 2bt + c \tag{2}$$

does not become zero for any real number t.

(ii) If $b^2 - ac = 0$, equation (1) has equal real roots $t = -b/a$. Expression (2) equals zero only if $t = -b/a$.

Using these facts, we immediately prove the following.

SCHWARZ INEQUALITY. *For any pair, y and y_1, of functions of x we have the inequality*

$$\left[\int_{x'}^{x''} yy_1 dx\right]^2 \leq \int_{x'}^{x''} y^2 dx \cdot \int_{x'}^{x''} y_1{}^2 dx, \tag{3}$$

in which equality holds if and only if the functions are proportional; that is,

$$y_1(x) = ky(x)$$

for some constant k.

Proof. For any real number t we have

$$\int_{x'}^{x''} (y + ty_1)^2\, dx = \int_{x'}^{x''} y^2\, dx + 2t \int_{x'}^{x''} yy_1\, dx + t^2 \int_{x'}^{x''} y_1{}^2\, dx.$$

The expression in the integral on the left is the square of some function. Therefore the integral is nonnegative. This integral is zero if and only if

$$y(x) + ty_1(x) = 0$$

for every x between x' and x''. In this case

$$\frac{y(x)}{y_1(x)} = -t,$$

and the functions y and y_1 are proportional, with $-t$ as the coefficient of proportionality.

If the two functions are not proportional, then the integral on the left is positive for every t (assuming $x' < x''$). Letting

$$\int_{x'}^{x''} y^2\, dx = a, \qquad \int_{x'}^{x''} yy_1\, dx = b, \qquad \int_{x'}^{x''} y_1{}^2\, dx = c,$$

we see that if functions y and y_1 are not proportional, then the expression

$$a + 2bt + ct^2$$

is not zero for any real number t. Therefore, we have

$$b^2 - ac < 0.$$

But if the functions are proportional, then this expression becomes zero if and only if $-t$ is the constant ratio of these functions:

$$-t = \frac{y(x)}{y_1(x)}.$$

In this case

$$b^2 - ac = 0.$$

Thus we always have the inequality

$$b^2 - ac \leq 0, \quad \text{that is,} \quad b^2 \leq ac,$$

which is the inequality

$$\left[\int_{x'}^{x''} yy_1\,dx\right]^2 \leq \int_{x'}^{x''} y^2\,dx \cdot \int_{x'}^{x''} y_1{}^2\,dx,$$

in which equality holds if and only if the functions y and y_1 are proportional. The proof is complete.

Now let us prove some additional algebraic inequalities.
It is always true that

$$a^2 - 2ab + b^2 = (a - b)^2 \geq 0,$$

from which we obtain

$$a^2 + b^2 \geq 2ab. \tag{4}$$

We have proved:

THEOREM 7. *The sum of the squares of two real numbers, a and b, is not less than twice their product. Equality holds when $a = b$.*

THEOREM 8. *For four real numbers, a, a_1, b, and b_1, we have*

$$aa_1 - bb_1 \geq \sqrt{a^2 - b^2} \cdot \sqrt{a_1^2 - b_1^2} \tag{5}$$

if all differences are nonnegative. Equality holds when

$$ab_1 - a_1 b = 0.$$

Proof.

$$(aa_1 - bb_1)^2 = a^2 a_1^2 - 2aa_1 bb_1 + b^2 b_1^2,$$
$$(a^2 - b^2)(a_1^2 - b_1^2) = a^2 a_1^2 - a^2 b_1^2 - a_1^2 b^2 + b^2 b_1^2.$$

By inequality (4), the sum of the squares of the numbers ab_1 and $a_1 b$ is not less than twice their product:

$$a^2 b_1^2 + a_1^2 b^2 \geq 2aa_1 bb_1,$$

where equality holds if

$$ab_1 - a_1 b = 0.$$

From this it follows that

$$(aa_1 - bb_1)^2 \geq (a^2 - b^2)(a_1^2 - b_1^2),$$

and our theorem is proved.

THEOREM 9. *For any pair of functions y and y_1, with x as the independent variable, and any pair of real numbers, a and a_1, we have the inequality*

$$aa_1 - \int_{x'}^{x''} yy_1 \, dx$$
$$\geq \sqrt{\left(a^2 - \int_{x'}^{x''} y^2 \, dx\right)\left(a_1^2 - \int_{x'}^{x''} y_1^2 \, dx\right)} \tag{6}$$

if all differences are nonnegative. Equality holds if and only if the functions y and y_1 are proportional to the numbers a and a_1:

$$\frac{y_1(x)}{y(x)} = \frac{a_1}{a}.$$

Proof. By the Schwartz inequality,

$$\int_{x'}^{x''} yy_1\,dx \leq \sqrt{\int_{x'}^{x''} y^2\,dx \cdot \int_{x'}^{x''} y_1{}^2\,dx}. \tag{3'}$$

Therefore,

$$aa_1 - \int_{x'}^{x''} yy_1\,dx \geq aa_1 - \sqrt{\int_{x'}^{x''} y^2\,dx \cdot \int_{x'}^{x''} y_1{}^2\,dx}. \tag{7}$$

In inequality (5), let

$$b = \sqrt{\int_{x'}^{x''} y^2\,dx}, \qquad b_1 = \sqrt{\int_{x'}^{x''} y_1{}^2\,dx}, \tag{8}$$

obtaining

$$aa_1 - \sqrt{\int_{x'}^{x''} y^2\,dx \cdot \int_{x'}^{x''} y_1{}^2\,dx}$$

$$\geq \sqrt{\left(a^2 - \int_{x'}^{x''} y^2\,dx\right)\left(a_1{}^2 - \int_{x'}^{x''} y_1{}^2\,dx\right)}. \tag{9}$$

Inequality (6) follows from inequalities (7) and (9).

For equality to hold in (6) we must have equality in both (3′) and (9), and for this it is necessary that both

$$\frac{y_1(x)}{y(x)} = k,$$

where k is some constant, and

$$\frac{a_1}{a} = \frac{b_1}{b};$$

by (8),

$$\frac{a_1}{a} = \sqrt{\frac{\int_{x'}^{x''} y_1{}^2\,dx}{\int_{x'}^{x''} y^2\,dx}} = k.$$

30. RELATION BETWEEN AREAS OF $\mathfrak{Q}$, $\mathfrak{Q}_1$, AND $\mathfrak{Q}_s$; THE BRUNN-MINKOWSKI INEQUALITY

BRUNN-MINKOWSKI THEOREM. *Suppose that we are given plane convex figures*

$$\mathfrak{Q},\quad \mathfrak{Q}_1,\quad \mathfrak{Q}_s = s\mathfrak{Q} + s_1\mathfrak{Q}_1 \quad (s + s_1 = 1;\ s, s_1 \geq 0),$$

and denote their areas by $J(\mathfrak{Q})$, $J(\mathfrak{Q}_1)$, *and* $J(\mathfrak{Q}_s)$. *Then we have the inequality*

$$\sqrt{J(\mathfrak{Q}_s)} \geq s\sqrt{J(\mathfrak{Q})} + s_1\sqrt{J(\mathfrak{Q}_1)} \tag{1}$$

in which equality holds if and only if $\mathfrak{Q}$ *and* $\mathfrak{Q}_1$ *are homothetic, that is, similar and similarly arranged.*

Inequality (1) was first proved by G. Brunn. G. Minkowski determined when it becomes an equality. We remark that inequality (1) is valid not only for convex figures but also for arbitrary figures.

Proof. We shall prove first that if $\mathfrak{Q}$ and $\mathfrak{Q}_1$ are homothetic, the equality holds in (1). If $\mathfrak{Q}$ and $\mathfrak{Q}_1$ are homothetic, then $\mathfrak{Q}_s$ is homothetic to both of them. Thus if $\mathfrak{Q}_1$ is congruent to $k\mathfrak{Q}$, then $\mathfrak{Q}_s$ is congruent to

$$s\mathfrak{Q} + ks_1\mathfrak{Q} = (s + s_1k)\mathfrak{Q}.$$

Under these conditions

$$J(\mathfrak{Q}_1) = k^2J(\mathfrak{Q}), \quad J(\mathfrak{Q}_s) = (s + s_1k)^2J(\mathfrak{Q}),$$
$$\begin{aligned}\sqrt{J(\mathfrak{Q}_s)} = (s + s_1k)\sqrt{J(\mathfrak{Q})} &= s\sqrt{J(\mathfrak{Q})} + s_1k\sqrt{J(\mathfrak{Q})}\\ &= s\sqrt{J(\mathfrak{Q})} + s_1\sqrt{J(\mathfrak{Q}_1)}.\end{aligned}$$

Thus, we see that formula (1) is satisfied for homothetic figures, and equality holds.

We shall show that inequality (1) is equivalent to the following inequality

$$J(\mathfrak{Q}, \mathfrak{Q}_1) \geq \sqrt{J(\mathfrak{Q})\cdot J(\mathfrak{Q}_1)}. \tag{2}$$

For the proof, recall formula (5′) of section 27:

$$J(\mathfrak{Q}_s) = s^2J(\mathfrak{Q}) + 2ss_1J(\mathfrak{Q}, \mathfrak{Q}_1) + s_1{}^2J(\mathfrak{Q}_1). \tag{3}$$

By squaring both sides of inequality (1), we obtain

$$J(\mathfrak{Q}_s) \geq s^2J(\mathfrak{Q}) + 2ss_1\sqrt{J(\mathfrak{Q})\cdot J(\mathfrak{Q}_1)} + s_1{}^2J(\mathfrak{Q}_1). \tag{4}$$

By means of formula (3), inequality (4) may be obtained from (2) and conversely. We shall undertake to prove (2) and hence (1).

There are many proofs of the Brunn-Minkowski theorem. We shall use the geometric method of Frobenius (1849–1917), a German mathematician.

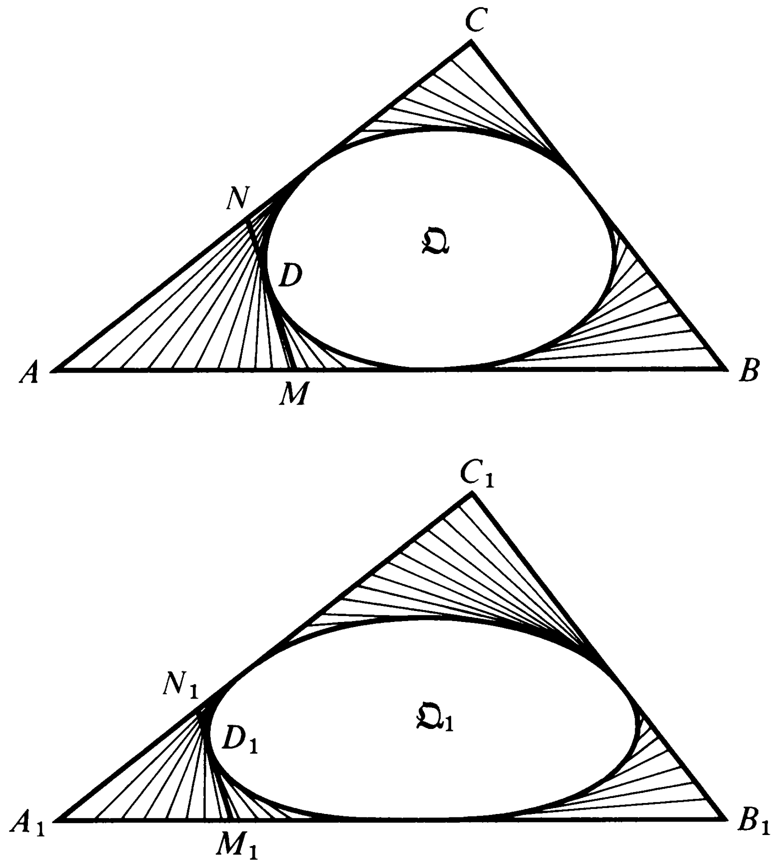


Fig. 119

Consider two arbitrary convex figures $\mathfrak{Q}$ and $\mathfrak{Q}_1$ (Fig. 119). Around these figures construct circumscribed triangles ABC and $A_1B_1C_1$ with corresponding sides parallel. If

$$\mathfrak{Q}_s = s\mathfrak{Q} + s_1\mathfrak{Q}_1 \quad (s + s_1 = 1;\ s, s_1 \geq 0),$$

then

$$\triangle A_sB_sC_s = s\triangle ABC + s_1\triangle A_1B_1C_1.$$

Here, $\triangle A_sB_sC_s$ is the triangle circumscribed about $\mathfrak{Q}_s$ with sides parallel to the sides of the other two triangles (and therefore similar to both of them).

Through a point D on the boundary of figure $\mathfrak{Q}$ draw a supporting line. It intersects triangle ABC at points M and N. Segment DM, having the direction of positive motion around $\mathfrak{Q}$, is called a *positive*

supporting segment, and DN, a *negative supporting segment.* Let D_1M_1 and D_sM_s be the positive supporting segments for figures $\mathfrak{Q}_1$ and $\mathfrak{Q}_s$ parallel to DM. These segments make some angle α with the x-axis. We denote the length of DM by $l(\alpha)$, the length of D_1M_1 by $l_1(\alpha)$, and the length of D_sM_s by $l_s(\alpha)$. We have

$$l_s(\alpha) = sl(\alpha) + s_1l_1(\alpha). \tag{5}$$

Consider the set of all positive supporting segments for figure $\mathfrak{Q}$ (also for $\mathfrak{Q}_1$ and $\mathfrak{Q}_s$). It fills the parts of triangle ABC that lie outside $\mathfrak{Q}$. We denote it by $(\triangle ABC - \mathfrak{Q})$. (Analogously, the positive supporting segments for figures $\mathfrak{Q}_1$ and $\mathfrak{Q}_s$ fill the regions $(\triangle A_1B_1C_1 - \mathfrak{Q}_1)$ and $(\triangle A_sB_sC_s - \mathfrak{Q}_s)$).

We wish to find the area of region $(\triangle ABC - \mathfrak{Q})$. To do so, at point D on the boundary of $\mathfrak{Q}$ construct positive supporting segment DM. Draw a line DM' to some point M' very close to M on triangle ABC. We have constructed an infinitesimal triangle MDM', with an infinitesimal angle $d\alpha$ at vertex D. As before, $l(\alpha)$ denotes the length of DM. The area of this infinitesimal triangle is given by $\frac{1}{2}l^2(\alpha)\,d\alpha$.

The area of region $(\triangle ABC - \mathfrak{Q})$ may be considered as the sum of the areas of such infinitesimal triangles. We represent the area by the integral

$$\frac{1}{2}\int_0^{2\pi} l^2(\alpha)\,d\alpha.$$

Let us denote the area of triangle ABC by J, of triangle $A_1B_1C_1$ by J_1, and of triangle $A_sB_sC_s$ by J_s. Then we have

$$J(\mathfrak{Q}) = J - \frac{1}{2}\int_0^{2\pi} l^2(\alpha)\,d\alpha. \tag{6}$$

Analogously,

$$J(\mathfrak{Q}_1) = J_1 - \frac{1}{2}\int_0^{2\pi} l_1{}^2(\alpha)\,d\alpha, \tag{7}$$

and

$$J(\mathfrak{Q}_s) = J_s - \frac{1}{2}\int_0^{2\pi} l_s{}^2(\alpha)\,d\alpha. \tag{8}$$

Because triangles ABC, $A_1B_1C_1$, and $A_sB_sC_s$ are homothetic, we have

$$J_s = (s\sqrt{J} + s_1\sqrt{J_1})^2. \tag{9}$$

From formulas (9), (5), and (8), it follows that

$$
\begin{aligned}
J(\mathfrak{Q}_s) &= (s\sqrt{J} + s_1\sqrt{J_1})^2 - \frac{1}{2}\int_0^{2\pi}[sl(\alpha) + s_1 l_1(\alpha)]^2\,d\alpha \\
&= s^2 J + 2ss_1\sqrt{JJ_1} + s_1^2 J_1 \\
&\qquad - \frac{1}{2}\int_0^{2\pi}[s^2 l^2(\alpha) + 2ss_1 l(\alpha)l_1(\alpha) + s_1^2 l_1^2(\alpha)]\,d\alpha \\
&= s^2\left[J - \frac{1}{2}\int_0^{2\pi} l^2(\alpha)d\alpha\right] + 2ss_1\left[\sqrt{JJ_1} - \frac{1}{2}\int_0^{2\pi} l(\alpha)l_1(\alpha)\,d\alpha\right] \\
&\qquad + s_1^2\left[J_1 - \frac{1}{2}\int_0^{2\pi} l_1^2(\alpha)\,d\alpha\right].
\end{aligned}
$$

On the other hand (from formula (5′) of section 27),

$$J(\mathfrak{Q}_s) = s^2 J(\mathfrak{Q}) + 2ss_1 J(\mathfrak{Q}, \mathfrak{Q}_1) + s_1^2 J(\mathfrak{Q}_1),$$

in which the coefficient of s^2 is the area of $\mathfrak{Q}$, the coefficient of s_1^2 is the area of $\mathfrak{Q}_1$, and the coefficient of $2ss_1$ is the mixed area of $\mathfrak{Q}$ and $\mathfrak{Q}_1$. Therefore,

$$J(\mathfrak{Q}, \mathfrak{Q}_1) = \sqrt{JJ_1} - \frac{1}{2}\int_0^{2\pi} l(\alpha)\, l_1(\alpha)\, d\alpha. \tag{10}$$

Now turn to inequality (6) of section 29. Let the numbers a, a_1 be defined so that

$$J = a^2 \quad \text{and} \quad J_1 = a_1^2; \tag{11}$$

hence,

$$\sqrt{JJ_1} = aa_1.$$

Then replace $y(x)$, $y_1(x)$, and x in (6) of section 29 by $\dfrac{l(\alpha)}{\sqrt{2}}$, $\dfrac{l_1(\alpha)}{\sqrt{2}}$, and α. Thus,

$$
\begin{aligned}
&\sqrt{JJ_1} - \frac{1}{2}\int_0^{2\pi} l(\alpha)\, l_1(\alpha)\, d\alpha \\
&\qquad \geq \sqrt{\left[J - \frac{1}{2}\int_0^{2\pi} l^2(\alpha)\, d\alpha\right]\left[J_1 - \frac{1}{2}\int_0^{2\pi} l_1^2(\alpha)\, d\alpha\right]},
\end{aligned}
$$

and from (6), (7), and (10), it follows that

$$J(\mathfrak{Q}, \mathfrak{Q}_1) \geq \sqrt{J(\mathfrak{Q}) \cdot J(\mathfrak{Q}_1)}. \tag{12}$$

Thus, the Brunn-Minkowski inequality is proved.

It remains to be proved that if the equality in (12) holds, $\mathfrak{Q}$ and $\mathfrak{Q}_1$ are homothetic. For equality to hold in (12) it is necessary that for all α

$$\frac{l(\alpha)}{l_1(\alpha)} = \frac{\sqrt{J}}{\sqrt{J_1}}.$$

We shall show that this happens only when $\mathfrak{Q}$ and $\mathfrak{Q}_1$ are homothetic. Note that the ratio of similitude of the similar triangles ABC and $A_1B_1C_1$ is $\sqrt{J}/\sqrt{J_1}$. Denote this ratio by k. Then the ratio of the lengths of corresponding sides of these circumscribed triangles is k; the ratio of the lengths of parallel positive supporting segments for figures $\mathfrak{Q}$ and $\mathfrak{Q}_1$ is also k. In all our discussions, if we replace positive supporting segments by negative supporting segments, then we obtain the same result: the ratio of the lengths of parallel negative supporting segments is also equal to k.

Now let MN and M_1N_1 be segments of parallel supporting lines for $\mathfrak{Q}$ and $\mathfrak{Q}_1$ cut off by the boundaries of triangles ABC and $A_1B_1C_1$. We have

$$MN = MD + DN,$$
$$M_1N_1 = M_1D_1 + D_1N_1,$$

where MD and M_1D_1 are positive, and DN and D_1N_1 are negative supporting segments. Since

$$MD = kM_1D_1,$$
$$DN = kD_1N_1,$$

we have

$$MN = kM_1N_1.$$

Now consider the figure $\mathfrak{Q}_2 = k\mathfrak{Q}_1$; around it is the circumscribed triangle $\triangle A_2B_2C_2 = k\triangle A_1B_1C_1$.Segment $M_2N_2 = kM_1N_1$ is a supporting segment for $\mathfrak{Q}_2$ with end points lying on the boundary of triangle $A_2B_2C_2$. The corresponding sides of triangles ABC and $A_2B_2C_2$ are equal and parallel. Thus, the triangles are congruent. The triangles cut off of ABC and $A_2B_2C_2$ by segments MN and M_2N_2 (for example, triangles AMN and $A_2M_2N_2$) are also congruent. They are similar because their corresponding sides are parallel, and congruent because sides MN and $M_2N_2 = kM_1N_1$ are equal. Triangle $A_2B_2C_2$ is obtained from triangle ABC by a translation. The same translation applied to triangle AMN yields triangle $A_2M_2N_2$, and applied to supporting line MN, it yields M_2N_2. Since the pair of sup-

porting lines MN and M_2N_2 is arbitrary, we conclude that any supporting line for figure $\mathfrak{D}_2$ is obtained from a supporting line for figure $\mathfrak{D}$ by applying the same translation.

Since figures $\mathfrak{D}$ and $\mathfrak{D}_2 = k\mathfrak{D}_1$ are determined by their supporting lines, $\mathfrak{D}$ and $\mathfrak{D}_2$ must be related by the same translation as their supporting lines. Thus, $\mathfrak{D}$ is obtained from $\mathfrak{D}_1$ by multiplication by k followed by a translation; that is, figures $\mathfrak{D}$ and $\mathfrak{D}_1$ are homothetic.

This completes the proof that equality holds in the Brunn-Minkowski inequality only for homothetic figures.

Remark. The Brunn-Minkowski theorem is easily generalized to n-dimensional space. For example, for three-dimensional space, we replace the square roots in inequality (1) by cube roots and use volume instead of area.

31. RELATION BETWEEN AREAS OF PLANE SECTIONS OF CONVEX SOLIDS

Suppose that we are given two plane convex figures $\mathfrak{D}$ and $\mathfrak{D}_1$ in two parallel planes P and P_1 (Fig. 120). We can form a solid T by connecting every point q of figure $\mathfrak{D}$ with every point q_1 of figure $\mathfrak{D}_1$ with straight line segments. The set of all these segments forms some solid T. (In Figure 120, figures $\mathfrak{D}$ and $\mathfrak{D}_1$ are represented by parallelograms that are not similar.)

Consider the section $\mathfrak{D}_s$ of solid T made by plane $P_s = sP + s_1P_1$ ($s + s_1 = 1$; $s, s_1 \geq 0$). Plane P_s is parallel to the planes of figures $\mathfrak{D}$ and $\mathfrak{D}_1$ and divides the distance between them in the ratio s_1/s. Each segment qq_1 of solid T (q belongs to $\mathfrak{D}$, q_1 to $\mathfrak{D}_1$) intersects plane P_s at the point which divides the segment in the ratio s_1/s, that is, at the point

$$q_s = sq + s_1q_1.$$

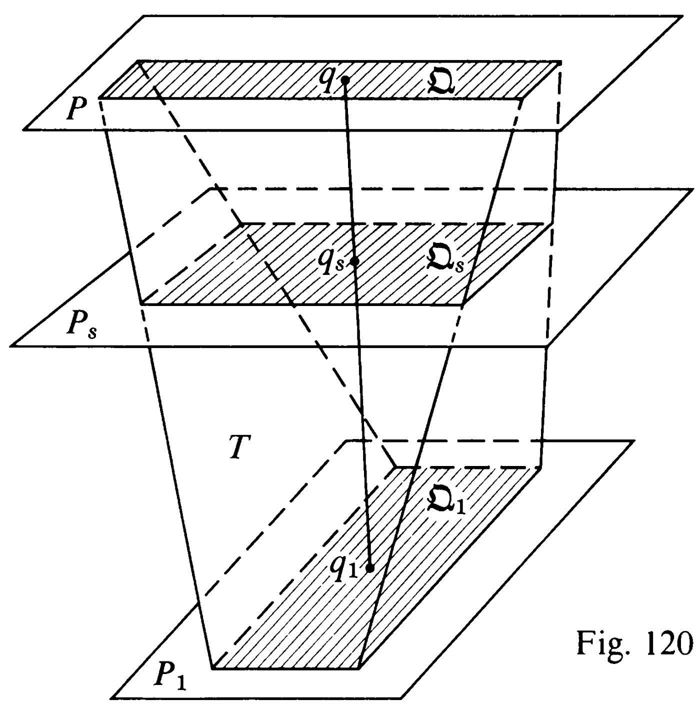


Fig. 120

Denoting the set of all points q_s by $\mathfrak{D}_s$, we have $\mathfrak{D}_s = s\mathfrak{D} + s_1\mathfrak{D}_1$.

In the preceding section we proved the Brunn-Minkowski theorem for figures $\mathfrak{Q}$, $\mathfrak{Q}_1$, and $\mathfrak{Q}_s$ lying in the same plane. But every step of the proof is equally correct for $\mathfrak{Q}$, $\mathfrak{Q}_1$, and $\mathfrak{Q}_s$ lying in different but parallel planes. Thus we still have the inequality

$$\sqrt{J(\mathfrak{Q}_s)} \geq s\sqrt{J(\mathfrak{Q})} + s_1\sqrt{J(\mathfrak{Q}_1)}. \tag{1}$$

Also, inequality (1) becomes an equality if and only if figures $\mathfrak{Q}$ and $\mathfrak{Q}_1$ are homothetic. If $\mathfrak{Q}$ and $\mathfrak{Q}_1$ are homothetic, then the solid T is easily seen to be either a cylinder (Fig. 121) or a frustrum of a cone (Fig. 122) (or simply a cone, if one of $\mathfrak{Q}$ and $\mathfrak{Q}_1$ is a point (Fig. 123)). For simplicity, we use circular cylinders and cones in our illustrations, but $\mathfrak{Q}$ and $\mathfrak{Q}_1$ may have other shapes.

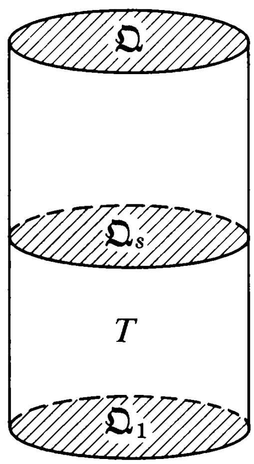


Fig. 121

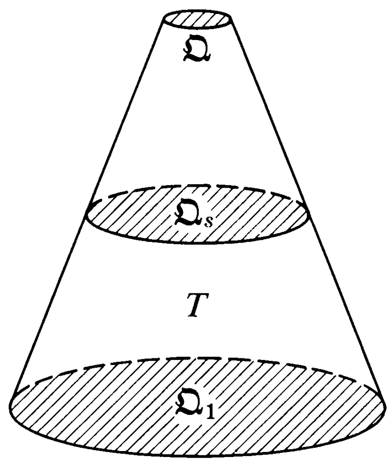


Fig. 122

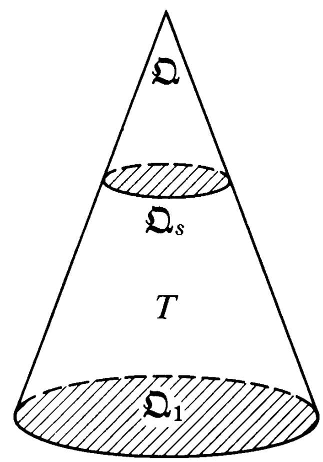


Fig. 123

Now suppose that we are given a convex solid K and some straight line which we shall consider as the x-axis. We shall study the sections of K made by planes perpendicular to this axis (Fig. 124).

Let us take two points x, x_1 on the x-axis and an arbitrary point x_s between them, such that

$$x_s = sx + s_1x_1 \ (s + s_1 = 1;\ s, s_1 \geq 0).$$

Denote by Q, Q_1, and Q_s the sections of solid K made by the planes P, P_1, and P_s perpendicular to the x-axis at points x, x_1, and x_s. Note that this definition of Q_s is different from the one given before.

THEOREM. *The areas of sections Q, Q_1, and Q_s as defined above satisfy the inequality.*

$$\sqrt{J(Q_s)} \geq s\sqrt{J(Q)} + s_1\sqrt{J(Q_1)}, \tag{2}$$

in which equality holds if and only if the part of solid K between planes P and P_1 is a cylinder or a frustrum of a cone.

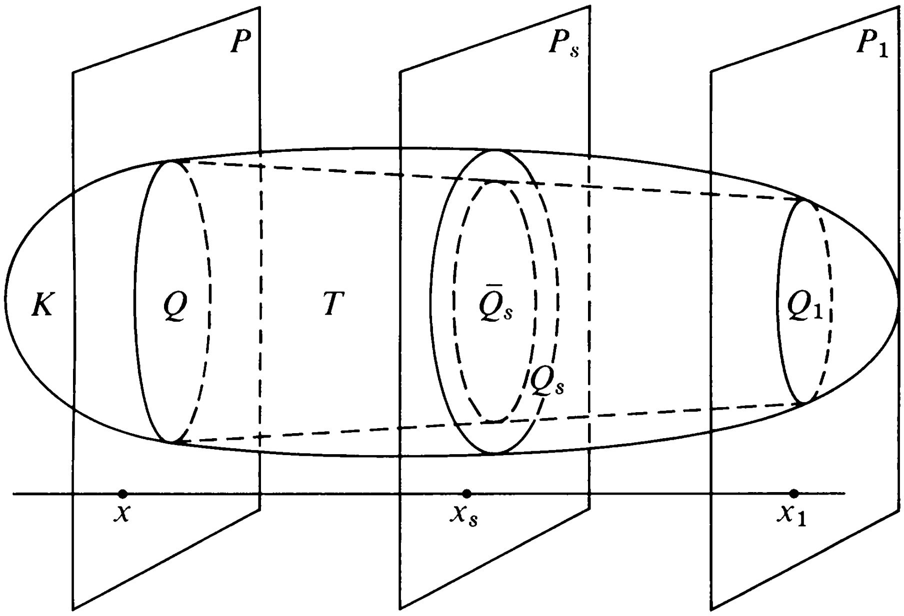


Fig. 124

Proof. Let T denote the figure composed of all line segments connecting points of section Q with section Q_1. Since Q and Q_1 are contained in convex solid K, all segments connecting their points belong to K. The solid T, composed of these segments, is contained in K. Let $\overline{Q}_s$ denote the section of T made by plane P_s. Since T is contained in K, then $\overline{Q}_s$ is contained in Q_s. Hence, the area of Q_s is greater than or equal to the area of $\overline{Q}_s$.

$$J(Q_s) \geq J(\overline{Q}_s).$$

The Brunn-Minkowski inequality is already proved for solid T:

$$\sqrt{J(\overline{Q}_s)} \geq s\sqrt{J(Q)} + s_1\sqrt{J(Q_1)}. \tag{3}$$

Therefore,

$$\sqrt{J(Q_s)} \geq s\sqrt{J(Q)} + s_1\sqrt{J(Q_1)}.$$

Equality holds in (2) if and only if both the following conditions are satisfied: (1) The part of solid K contained between planes P and P_1 coincides with solid T. (2) Sections Q and Q_1 are homothetic. These conditions are satisfied whenever the part of K contained between P and P_1 is a cyclinder or a frustrum of a cone, and the theorem is proved.

We also have the following:

COROLLARY. *If the sections of convex solid K made by parallel planes P, P_1, and P_s have equal areas, then the part of solid K between planes P and P_1 is a cylinder.*

Proof. The preceding theorem tells us that this part of K is either a cylinder or a frustrum of a cone. But since the areas of the sections are equal, it must be a cylinder.

32. GREATEST AREA THEOREMS

The Brunn-Minkowski theorem has some interesting consequences.

L'HUILIER'S THEOREM. *Of all convex polygons with sides having given directions and with perimeter of given length l, the one with the greatest area is circumscribed about a circle.*

Proof. Let Q be a polygon with perimeter of length l, and let Q_1 be a polygon with sides parallel to the sides of Q, and circumscribed about a circle with radius 1. Let S denote the area of Q and S_1 the area of Q_1. By the Brunn-Minkowski inequality (formula (2) of section 30), we have

$$J(Q, Q_1) \geq \sqrt{J(Q) \cdot J(Q_1)} = \sqrt{S \cdot S_1}, \qquad (1)$$

where equality holds if Q and Q_1 are homothetic.

The mixed area $J(Q, Q_1)$ of polygons Q and Q_1 is numerically equal to half the length of the perimeter of Q (section 28):

$$J(Q, Q_1) = \frac{1}{2}l.$$

From this and inequality (1) follow

$$\frac{1}{2}l \geq \sqrt{S \cdot S_1},$$

$$\frac{1}{4}l^2 \geq S \cdot S_1,$$

and

$$J(Q) = S \leq \frac{l^2}{4S_1}, \qquad (2)$$

in which if Q is not circumscribed about a circle (that is, not homothetic to Q_1), we have the strict inequality

$$J(Q) < \frac{l^2}{4S_1}.$$

But if Q is circumscribed about a circle, inequality (2) becomes the equation

$$J(Q) = \frac{l^2}{4S_1}.$$

Thus, the maximum area of polygon Q is $\frac{l^2}{4S_1}$ and is attained if Q is circumscribed about a circle (that is, is homothetic to Q_1).

THEOREM 11 (ISOPERIMETRIC PROBLEM). *Of all convex figures with boundary of given length l, a circle has the greatest area.*

Proof. Let $\mathfrak{Q}$ be a convex figure with boundary of length l and let Q_1 be a circle with radius 1. Denote the area of $\mathfrak{Q}$ by S. The area of Q_1 is π and its boundary has length 2π.

By the Brunn-Minkowski inequality we have

$$J(\mathfrak{Q}, Q_1) \geq \sqrt{J(\mathfrak{Q}) \cdot J(Q_1)} = \sqrt{\pi S},$$

which becomes a strict inequality if $\mathfrak{Q}$ is not a circle.

The mixed area $J(\mathfrak{Q}, Q_1)$ is numerically equal to half the length l of the boundary of $\mathfrak{Q}$ (section 28); so we have

$$\frac{1}{2}l \geq \sqrt{\pi S},$$

from which we obtain

$$S \leq \frac{l^2}{4\pi}. \tag{3}$$

Equality holds in (3) if and only if $\mathfrak{Q}$ is a circle. Thus, of all convex figures with boundary of length l, a circle has the greatest area.

We shall prove later (section 40) that inequality (3) holds for all (not only convex) figures and that the circle has the greatest area among *all* figures of given perimeter.

5. Theorems of Minkowski and Aleksandrov for Congruent Convex Polyhedra

33. FORMULATION OF THE THEOREMS

DEFINITION. *Two figures are said to be* congruent by translation *if one of them can be translated to coincide with the other (reflections and rotations are excluded).*

We have the following theorem about convex polygons. Its proof is easy enough to be left for the reader.

THEOREM 1. *If each side of one convex polygon is equal to the corresponding side with parallel external normal of a second convex polygon and conversely, then the two polygons are congruent by translation.*

The interesting question about this theorem is whether it can be generalized to polyhedra. We can formulate such a theorem about polyhedra, requiring faces with parallel external normals to be congruent, but it turns out to be sufficient to require only that the corresponding faces have equal areas. This theorem was proved by G. Minkowski in 1897. It reads as follows:

MINKOWSKI'S THEOREM. *If each face of one convex polyhedron is equal in area to the corresponding face with parallel external normal of a second convex polyhedron and conversely, then the two polyhedra are congruent by translation.*

This theorem is far from being as simple as the theorem about polygons. To clarify the situation, we shall give an example. Suppose that we have a right prism. It is easy to prove that a polyhedron whose faces have external normals parallel to the external normals of the given prism must also be a right prism. It is also easy to prove that these prisms are congruent by translation if the faces with parallel external normals have equal areas. (We leave this task to the reader.) Here the work is easier because two prisms with parallel external normals have the same structure. But this is not true for general polyhedra with parallel external normals; for example, a face of one polyhedron may be a triangle while the face of the other

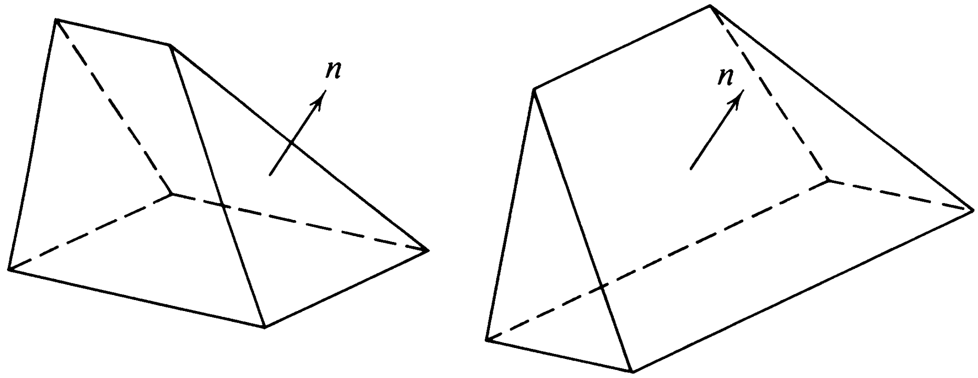


Fig. 125

with the parallel external normal may not be a triangle (Fig. 125). Notice that corresponding faces of these polyhedra have different areas. This situation can become complicated for polyhedra with a large number of faces. If we clearly understand this circumstance, we can begin to see the intrinsic difficulty of Minkowski's theorem in contrast to the analogous theorem for polygons.

Instead of proving Minkowski's theorem directly, we shall prove a more general theorem due to Aleksandrov, still using elementary methods. We shall need the following:

DEFINITION. *Polygon P_1 is said to be* imbedded inside *polygon P_2 if P_1 does not extend past the boundary of P_2, and at least one of its sides (with the possible exception of its end points) lies in the interior of P_2.*

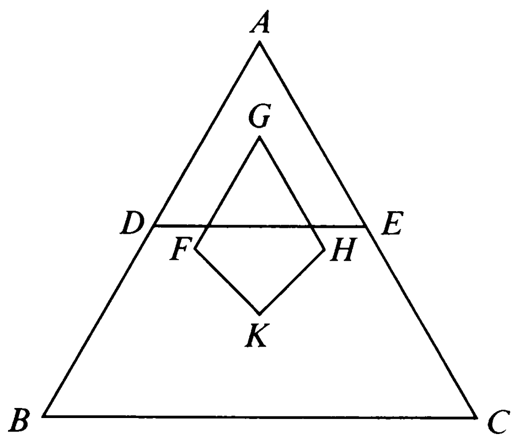


Fig. 126

In Figure 126 trapezoid $BCED$ and quadrilateral $FKHG$ are imbedded inside triangle ABC.

ALEKSANDROV'S THEOREM. *Let there be given two convex polyhedra such that to each face of one there corresponds a face of the other with a parallel external normal and conversely. If each pair of corresponding faces has the property that neither face can be imbedded inside the other by a translation, then the polyhedra are congruent by translation.*

In particular, suppose that we have two convex polyhedra H_1 and H_2. Let each face $P_1^{(k)}$ of polyhedron H_1 correspond to a face

$P_2^{(k)}$ of polyhedron H_2 with parallel external normals, so that the faces of H_1 and H_2 correspond in pairs $P_1^{(1)}$ and $P_2^{(1)}$, $P_1^{(2)}$ and $P_2^{(2)}$, and so on.

Suppose that each pair of faces $P_1^{(k)}$ and $P_2^{(k)}$ with parallel external normals has the property that $P_1^{(k)}$ cannot be imbedded inside $P_2^{(k)}$, and $P_2^{(k)}$ cannot be imbedded inside $P_1^{(k)}$. Then the theorem asserts that polyhedra H_1 and H_2 are congruent by translation. Hence, the corresponding faces $P_1^{(k)}$ and $P_2^{(k)}$ are also congruent by the same translation. Consequently, it is not possible for corresponding faces $P_1^{(k)}$ and $P_2^{(k)}$ to extend past each other. Thus the theorem asserts that if no face can be imbedded inside the corresponding face, then these faces can be made to coincide.

We shall show that the above theorem of Aleksandrov implies the theorem of Minkowski.

Proof of Minkowski's theorem. Suppose that convex polyhedra H_1 and H_2 have equal (in area) corresponding faces with parallel external normals. Equal faces obviously cannot be imbedded inside each other. Consequently, we see that the conditions of Minkowski's theorem imply the conditions of Aleksandrov's theorem, and if the latter is valid, then polyhedra H_1 and H_2 are congruent by translation.

The reader can easily see for himself how the following is obtained from Aleksandrov's theorem.

COROLLARY. *If corresponding faces of two convex polyhedra have parallel external normals and equal perimeters, then the polyhedra are congruent by translation.*

Before we give the proof of Aleksandrov's theorem, we shall first prove a theorem about convex polygons which we shall need, and second, consider a construction (construction of the mean polyhedron) which is important not only for this proof, but also in a number of other problems.

34. A THEOREM ABOUT CONVEX POLYGONS

We shall say that polygon P_1 is *imbedded in* P_2 either if it is imbedded inside P_2 (as defined in the preceding section) or if it coincides with P_2.

We shall compare the sides of polygons having parallel external normals, agreeing that if polygon P_1 has no side with an external normal parallel to the external normal to some side of polygon P_2,

then we shall say that *such a side exists but has length zero.* The reader must remember this convention (cf. section 27).

Let us agree to say that sides $l_1, l_2, \ldots, l_n$ are "greater than" sides $l_1', l_2', \ldots, l_n'$, if $l_1 \geq l_1'$, $l_2 \geq l_2'$, $\ldots$, $l_n \geq l_n'$ and if for at least one pair $l_i > l_i'$. (It is understood that only sides having parallel external normals are to be compared.) Analogously, we shall say that sides $l_1', \ldots, l_n'$ are "smaller than" sides $l_1, \ldots, l_n$.

Since we shall discuss only convex polygons and polyhedra in this section, we may sometimes omit the word "convex" from our discussion, but it always applies.

LEMMA 1. *If the sides of a polygon P_1, with the possible exception of one side l_0, are smaller than the sides of P_2, then P_1 can be imbedded inside P_2 by a translation.*

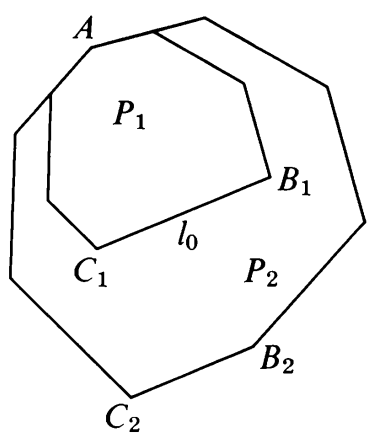


Fig. 127

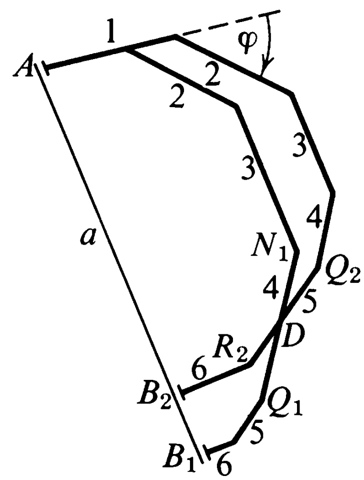


Fig. 128

Proof. Let A_1 be a vertex of P_1 through which passes a supporting line parallel to l_0 on the opposite side of P_1 from l_0. Let A_2 be the corresponding vertex of P_2. (The sides of P_1 and P_2 meeting in A_1 and A_2 must have parallel external normals.)

Translate P_1 so that A_1 coincides with A_2; $A_1 = A_2 = A$ in Figure 127. We shall show that this places P_1 inside P_2. The removal of vertex A and side l_0 divides the boundary of P_1 into two broken lines AB_1 and AC_1. Similarly, the removal of A and side B_2C_2 (B_2C_2 is parallel to l_0) divides the boundary of P_2 into broken lines AB_2 and AC_2.

The broken line AB_1 cannot pass outside polygon P_2 through AB_2. To prove this, let us suppose that AB_1 passes out of P_2 at a point D of AB_2 (Fig. 128). We draw a straight line a perpendicular to l_0. Going along broken lines AB_1 and AB_2, we number the sides

so that sides with parallel external normals are given the same number (remember the convention about sides of length zero). Between each side and the next is an angle, such as angle φ in Figure 128. Because of the parallel external normals, the angle between a pair of sides of AB_1 is equal to the angle between the sides of AB_2 having the same numbers. Suppose that point D lies on side N_1Q_1 of AB_1 and side Q_2R_2 of AB_2. The number assigned to N_1Q_1 must be smaller than the number assigned to Q_2R_2 because the passage from the side numbered 1 to side N_1Q_1 involves rotation through a smaller angle than does passage to Q_2R_2. Therefore the projection of AQ_1 on line a must be longer than the projection of AQ_2. (We reach the same conclusion in the case illustrated in Figure 129.)

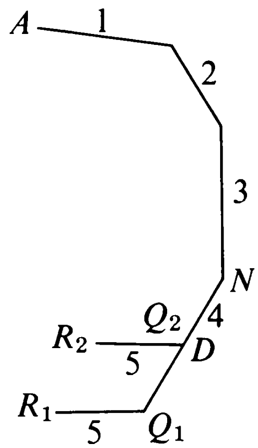


Fig. 129

But since the corresponding sides of these broken lines are parallel, then AQ_1, having the longer projection, must have a side longer than the corresponding side of AQ_2. (This can be seen in Figure 128.) But the conditions of our Lemma do not allow AB_1 to have a side longer than the corresponding side of AB_2. This contradiction proves that AB_1 cannot pass outside P_2 through AB_2.

In the same way we can prove that AC_1 cannot pass outside P_2 through AC_2.

Finally, neither AB_1 nor AC_1 can pass outside P_2 through a side corresponding to l_0. If it did, then the projection of it on line a would be longer than the projection of AB_2 or AC_2, which again implies that P_1 has a side other than l_0 that is longer than the corresponding side of P_2.

Thus, vertices B_1 and C_1 of P_1 lie in P_2, and so must the side l_0 because l_0 is B_1C_1. Hence P_1 cannot extend past P_2. Neither can it

coincide with P_2 because it has at least one side shorter than the corresponding side of P_2. Therefore P_1 is imbedded inside P_2 and Lemma 1 is proved.

LEMMA 2. *Suppose that two broken lines Q_1 and Q_2, lying in an angle with vertex O, have their end points on the sides of this angle and are convex toward O. If some half line from O meets Q_1 before Q_2, then Q_1 has a side shorter than the corresponding side of Q_2 with parallel external normal* (Fig. 130). (Remember our convention about sides of length zero.)

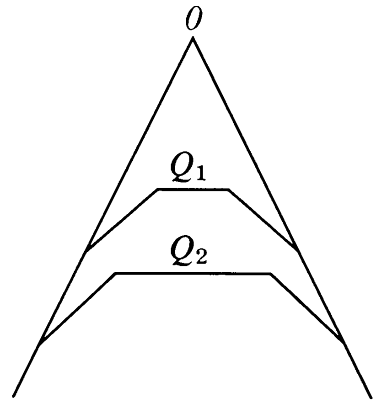


Fig. 130

Proof. For the proof we contract Q_2 toward O until it lies entirely in the region bounded by Q_1 of the angle with vertex at O. This contraction shortens the sides of Q_2. If after this contraction Q_1 and Q_2 coincide, then the sides of Q_2 must have been longer than the corresponding sides of Q_1, and the Lemma is proved. But if Q_1 and Q_2 do not coincide after the contraction, we must ask how they can fail to coincide. Perhaps, as shown in Figure 131, a side of Q_1 (the lower broken line) lies on a side of Q_2 that is longer, which gives us the desired result. Otherwise, Q_1 and Q_2 have a point (and perhaps one or more sides) in common (Fig. 132), but the sides of Q_1 and Q_2 terminating in this point do not coincide. Then these sides do not have parallel external normals. In fact, the side of Q_2 in question corresponds to a side of Q_1 with length zero. The Lemma is proved.

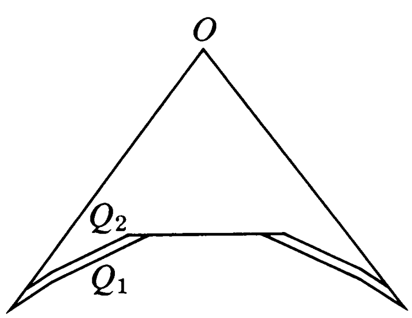


Fig. 131

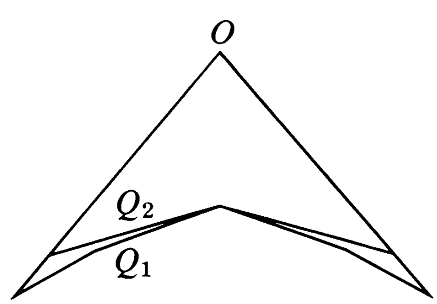


Fig. 132

Now we shall prove a theorem about polygons which we shall use in proving Aleksandrov's theorem about polyhedra. This Theorem is an analog of Lemma 2, section 20.

THEOREM 2. *If neither of two convex polygons can be imbedded in the other by a translation, then the difference between the lengths of their sides with parallel external normals changes sign at least four times in making one circuit around either polygon.*

In other words, if P_1 and P_2 are two polygons (Fig. 133), we make a plus by each side of polygon P_1 that is longer than the side of P_2 with parallel external normal and a minus by each side of P_1 that is shorter than the corresponding side of P_2 (remember the convention about sides of length zero). If the sides in question are equal, we make no mark. Then Theorem 2 says that we must encounter at least four changes of sign in making one circuit around P_1.

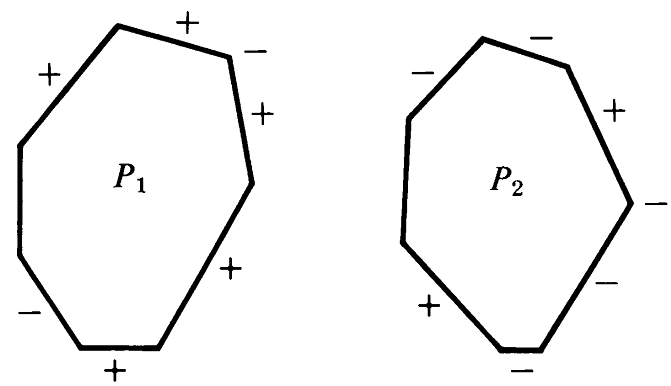


Fig. 133

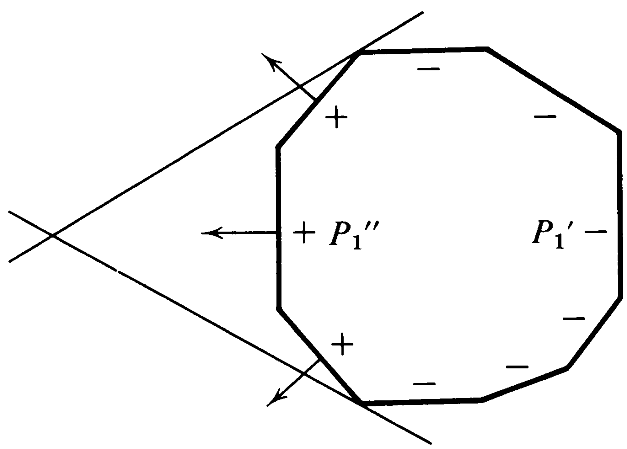
Proof. Let P_1 and P_2 be convex polygons, neither of which can be imbedded in the other by a translation. If the sides of P_1 were "smaller than" the sides of P_2 (as defined on page 135), then Lemma 1 would tell us that P_1 could be imbedded in P_2 by a translation, contradicting the conditions of the theorem. Similarly, if the sides of P_1 were greater than the sides of P_2, it would be possible to imbed P_2 in P_1. Consequently, both P_1 and P_2 have sides marked by plus signs and by minus signs. (The sides of P_1 and P_2 are to be marked as explained for P_1 in the preceding paragraph.) Hence, the difference between the lengths of the sides of P_1 and P_2 changes sign at least twice. (Notice that the number is always even; so if it is not zero, it must be at least two.)

Let us assume, contradicting the theorem, that there are only two changes of sign. Then each of polygons P_1 and P_2 could be divided into two broken lines P_1', P_1'' and P_2', P_2'' so that the sides of P_1' are smaller than the sides of P_2' and the sides of P_1'' are greater than

the sides of P_2''. The sides of P_1' and P_2', and of P_1'' and P_2'', have parallel external normals.

The sum of the angles between external normals to neighboring sides of a convex polygon is 2π (because the angle between external normals to neighboring sides is equal to the exterior angle between these sides, and the sum of the exterior angles is 2π). In dividing polygon P_1 into broken lines P_1' and P_1'', we eliminate the two angles between the external normals to the adjacent sides of P_1' and P_1''. Consequently, for at least one of these broken lines, the sum of the angles between the external normals is less than π. We shall assume that this is true for P_1'', and hence for P_2'', since their sides have parallel external normals. Then if we draw any two supporting lines for P_1'', P_1'' will be convex toward the vertex of the angle formed by these lines (Fig. 134). On the other hand, if the sum of

Fig. 134



the angles between the external normals to the sides of P_1'' is really greater than π, then this sum is less than π for P_1'. Then, interchanging the symbols for polygons P_1 and P_2 (it does not matter which is called the first and which the second), we get the sum of the angles between the external normals to P_1'' to be less than π and the sides of P_1'' to be greater than those of P_2''.

The sides of P_1' are smaller than those of P_2'. If we connect the end points of P_1' with a line segment and do the same for P_2', we obtain two convex polygons. By Lemma 1 the first of these can be imbedded inside the second by a translation. But by the conditions of the present theorem, after applying this translation to P_1, P_1'' must pass outside P_2. Then there is a point on P_1'' through which there passes a supporting line for P_1 not intersecting P_2. (Since P_1 extends outside P_2, P_2 has some supporting line b intersecting P_1. We can draw a supporting line a for P_1 at A parallel to b and lying on the opposite side of b from P_2. Line a satisfies our requirements.)

Moving this point along the boundary of P_1 and turning the supporting line first to the left and then to the right, we obtain two lines a_1 and a_2 which are supporting for both P_1 and P_2 (Fig. 135). These

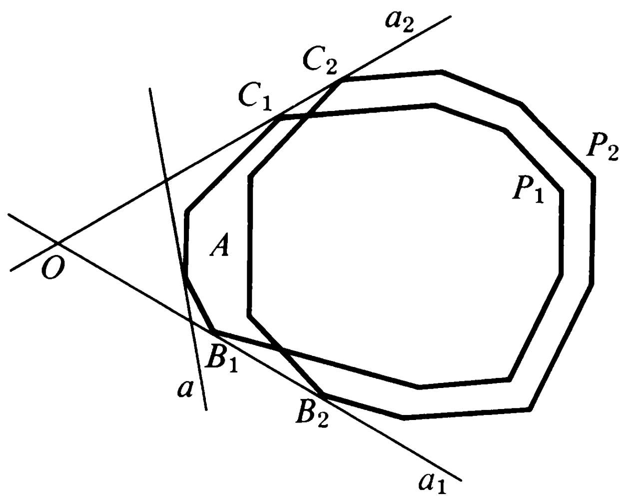


Fig. 135

lines touch P_1 at vertices B_1 and C_1, and they touch P_2 at vertices B_2 and C_2. (It could happen that one or both of these lines is tangent along a side of a polygon. Then we take the end point of this side that is closest to A.) Vertices B_1 and C_1 belong to P_1'' because P_1' lies inside P_2. Corresponding sides of P_2'' and P_1'' are parallel, and hence so are their supporting lines. Therefore a_1 and a_2 are supporting lines for P_2'' at vertices B_2 and C_2. (Observe that the validity of our discussions is unaffected by the existence of sides with length zero.)

Now we have the following situation: Broken lines B_1C_1 and B_2C_2 (parts of P_1'' and P_2'') have their end points on lines a_1 and a_2, which form an angle with vertex O. Both of these broken lines are convex toward O, since the sum of the angles between the external normals to P_1'' (also for P_2'') is less than π, so that the total angle of rotation of supporting line a from position a_1 to position a_2 is less than π.

Broken line B_1C_1 is closer to O than B_2C_2 is. This is easy to prove. From O draw a half-line through point A. This half-line will intersect B_2C_2 in a point farther from O than A because supporting line a at point A does not intersect P_2.

Applying Lemma 2, we see that B_1C_1 has a side shorter than the corresponding side of B_2C_2. Hence P_1'' has a side shorter than the corresponding side of P_2'', as is shown in our drawing. This contradicts our assumption that the sides of P_1'' are greater than the sides of P_2''. Consequently, the assumption that the difference between the lengths of corresponding sides of polygons P_1 and P_2 changes sign only twice is false. The number of changes of sign must be greater than two. But since it is even, then there must be at least four changes of sign, and the theorem is proved.

35. MEAN POLYGONS AND POLYHEDRA

In Chapter 4 we studied the operation of mixing figures. For simplicity we now restrict ourselves to the particular case of this operation where $s = s_1 = \frac{1}{2}$ (although all the following results are also valid for arbitrary s, where $0 < s < 1$).

Suppose that we are given two figures (plane or solid) H_1 and H_2. We can form the figure

$$H_{\frac{1}{2}} = \frac{1}{2}(H_1 + H_2),$$

the geometric locus of the midpoints of all line segments connecting points of H_1 with points of H_2.

DEFINITION. *The figure* $H_{\frac{1}{2}} = \frac{1}{2}(H_1 + H_2)$ *is called the* mean figure *of* H_1 *and* H_2.

We recall the following properties of the mixing operation:

(1) If H_1 and H_2 are convex figures, then $H_{\frac{1}{2}}$ is also a convex figure (Theorem 1, section 26).

(2) If Q_1 and Q_2 are two parallel planes, then $Q_{\frac{1}{2}} = \frac{1}{2}(Q_1 + Q_2)$ is a plane parallel to both of them, lying halfway between them (page 102).

(3) If one of the figures H_1 or H_2 is translated, then $H_{\frac{1}{2}}$ is also translated (Theorem 3, section 26).

Now we shall prove a theorem analogous to Theorem 2 of section 26.

THEOREM 3. *If H_1 and H_2 are two convex solids and Q_1 and Q_2 are supporting planes for them with parallel external normals, then the mean plane, $Q_{\frac{1}{2}} = \frac{1}{2}(Q_1 + Q_2)$, is a supporting plane for the mean solid, $H_{\frac{1}{2}} = \frac{1}{2}(H_1 + H_2)$, and its external normal is parallel to those of Q_1 and Q_2.*

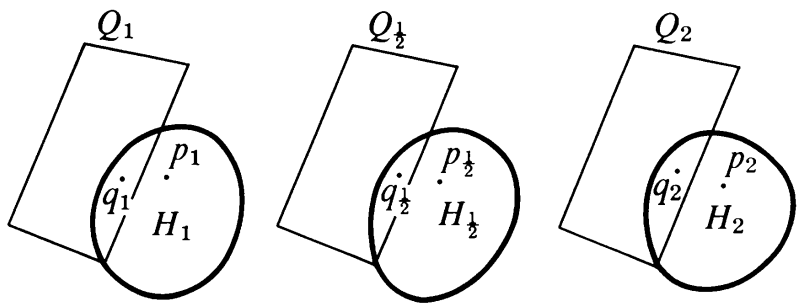


Fig. 136

Proof. Solids H_1 and H_2 lie on the same side of planes Q_1 and Q_2 (Fig. 136). We choose any pair of points p_1 and p_2 belonging to H_1 and H_2. The midpoint of line segment p_1p_2 is always on the same side of plane $Q_{\frac{1}{2}} = \frac{1}{2}(Q_1 + Q_2)$. In particular, if we draw planes P_1 and P_2 through p_1 and p_2 parallel to Q_1 and Q_2, then these planes lie on the same side of Q_1 and Q_2. Consequently, the mean plane $P_{\frac{1}{2}} = \frac{1}{2}(P_1 + P_2)$, on which lies the midpoint of segment p_1p_2, lies on the same side of the mean plane $Q_{\frac{1}{2}} = \frac{1}{2}(Q_1 + Q_2)$. Hence, the entire solid $H_{\frac{1}{2}}$ lies on one side of plane $Q_{\frac{1}{2}}$.

If q_1 and q_2 are points of solids H_1 and H_2 lying on the supporting planes Q_1 and Q_2, then point $q_{\frac{1}{2}}$ of solid $H_{\frac{1}{2}}$, the midpoint of segment q_1q_2, lies on the plane $Q_{\frac{1}{2}} = \frac{1}{2}(Q_1 + Q_2)$. Consequently, plane $Q_{\frac{1}{2}}$ has a point in common with solid $H_{\frac{1}{2}}$. And since the solid lies on one side of the plane, the plane is a supporting plane.

Finally, plane $Q_{\frac{1}{2}}$ is parallel to planes Q_1 and Q_2. Since solid $H_{\frac{1}{2}}$ lies on the same side of it as H_1 and H_2 of Q_1 and Q_2, we conclude that the external normal to plane $Q_{\frac{1}{2}}$ is parallel to those of Q_1 and Q_2. Thus, the theorem is proved.

THEOREM 4. *If P_1 and P_2 are two convex polygons lying in parallel planes, then the mean polygon, $P_{\frac{1}{2}} = \frac{1}{2}(P_1 + P_2)$, is a convex polygon lying in the mean plane between the planes of polygons P_1 and P_2.*

Proof. That $P_{\frac{1}{2}} = \frac{1}{2}(P_1 + P_2)$ lies in a mean plane follows from a fact we have used before: the midpoints of line segments connecting points of two parallel planes lie in a plane parallel to and halfway between the given planes.

If P_1 and P_2 lie in one plane, then $P_{\frac{1}{2}} = \frac{1}{2}(P_1 + P_2)$ is a convex polygon, by virtue of Theorem 1, section 26. From Theorem 3, section 26, it follows that if P_1 is translated to a parallel plane, then $P_{\frac{1}{2}}$ also undergoes only a translation and consequently remains a convex polygon.

Supplement. A line segment or point may be considered as a degenerate convex polygon. Hence we may admit the following cases:

(1) P_1 and P_2 are ordinary polygons.

(2) P_1 is a polygon and P_2 is a point (or conversely). Then $P_{\frac{1}{2}} = \frac{1}{2}(P_1 + P_2)$ is also a convex polygon, which can easily be shown to be similar to P_1 with sides half as long.

(3) P_1 is a polygon and P_2 is a line segment (or conversely). Then $P_{\frac{1}{2}} = \frac{1}{2}(P_1 + P_2)$ is also a convex polygon.

(4) P_1 and P_2 are line segments. If they are parallel, then, as shown in Figure 137, $P_{\frac{1}{2}} = \frac{1}{2}(P_1 + P_2)$ is a line segment parallel

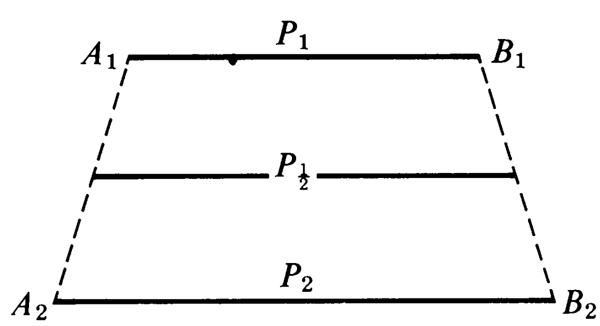


Fig. 137

to them and having the average of their lengths (cf. Example 3, section 26). In Figure 137 $P_{\frac{1}{2}}$ is the center line of a trapezoid with bases P_1 and P_2.

If P_1 and P_2 are two nonparallel segments, then $P_{\frac{1}{2}} = \frac{1}{2}(P_1 + P_2)$ is a parallelogram with sides parallel to P_1 and P_2 with the lengths of $\frac{1}{2}P_1$ and $\frac{1}{2}P_2$ respectively.

To prove this, let A_1 and B_1 be the end points of segment P_1, and A_2 and B_2 the end points of segment P_2 (Fig. 138). In mixing

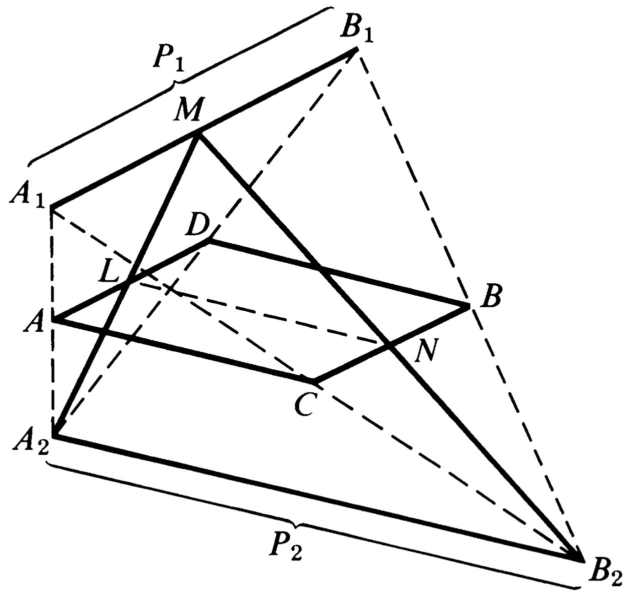


Fig. 138

P_1 and P_2 we must connect every point of P_1 with every point of P_2 and take the midpoints of the segments obtained. Let us first take the end point A_1 of segment P_1, connect it with all points of segment P_2, and take the midpoints of all the segments obtained. They form center line AC of triangle $A_1A_2B_2$. In exactly the same way, connecting B_1 with all points of P_2 and taking the midpoints of the segments produced, we obtain center line BD of triangle $B_1A_2B_2$. Also, AD and CB are center lines of triangles $A_2A_1B_1$ and $B_2A_1B_1$. AD and CB are parallel to A_1B_1, and AC and BD are parallel to A_2B_2. Consequently, $ACBD$ is a parallelogram. Its sides are equal to half of segments P_1 and P_2, since AC is equal to half of P_2 as a center line of triangle $A_1A_2B_2$ with base $A_2B_2 = P_2$ and AD is equal to half of P_1 as a center line of triangle $A_2A_1B_1$ with base $A_1B_1 = P_1$.

Now let us take an arbitrary point M on segment P_1. Connecting it with all points of segment P_2 and taking the midpoints of the segments thus formed, we obtain center line LN of triangle MA_2B_2.

Point L lies on side AD of our parallelogram. Because AD is a center line of triangle $A_2A_1B_1$, it bisects segment MA_2. Point L is the midpoint of this segment and hence lies on AD. For the same reason point N lies on segment CB. If we move point M from A_1 to B_1 along the entire segment P_1, then segment LN, the end points of which slide along AD and CB, will sweep out the entire parallelogram $ACBD$. We have shown that $P_{\frac{1}{2}} = \frac{1}{2}(P_1 + P_2)$ is this parallelogram. This proof is valid even if P_1 and P_2 do not lie in the same plane.

Now we shall prove a basic theorem about mean polyhedra, which will be used to prove Aleksandrov's theorem.

THEOREM 5. *If H_1 and H_2 are convex polyhedra, then $H_{\frac{1}{2}} = \frac{1}{2}(H_1 + H_2)$ is also a convex polyhedron. Its faces are obtained*

(1) *by mixing faces of polyhedra H_1 and H_2,*

(2) *by mixing a face of one of these polyhedra with an edge or a vertex of the other, and*

(3) *by mixing nonparallel edges of these polyhedra where the faces, edges, and vertices being mixed lie in supporting planes with parallel external normals.*

The edges of polyhedron $H_{\frac{1}{2}}$ are obtained

(1) *by mixing pairs of parallel edges, and*

(2) *by mixing edges with vertices lying in supporting planes with parallel external normals.*

Proof. Let H_1 and H_2 be convex polyhedra and $H_{\frac{1}{2}} = \frac{1}{2}(H_1 + H_2)$.

Let $Q_{\frac{1}{2}}$ be a supporting plane for $H_{\frac{1}{2}}$. From Theorem 3 it follows that $Q_{\frac{1}{2}}$ lies half way between two supporting planes Q_1 and Q_2 for H_1 and H_2 with parallel external normals.

If a face (edge, vertex) P_1 of polyhedron H_1 lies on Q_1 and a face (edge, vertex) P_2 of polyhedron H_2 lies on Q_2, then

$$P_{\frac{1}{2}} = \frac{1}{2}(P_1 + P_2)$$

will lie on $Q_{\frac{1}{2}}$, since the midpoints of line segments connecting points of Q_1 and Q_2 lie on $Q_{\frac{1}{2}}$. A face and a face, edge, or vertex give a face. An edge and a nonparallel edge give a face. An edge and a parallel edge or a vertex give an edge. A vertex and another vertex give a point.

Thus we see that there are plane faces on $H_{\frac{1}{2}}$. We shall show that no part of the surface of $H_{\frac{1}{2}}$ can be curved. We know that every point $P_{\frac{1}{2}}$ on the boundary of $H_{\frac{1}{2}}$ has the form

$$P_{\frac{1}{2}} = \frac{1}{2}(P_1 + P_2),$$

where P_1 and P_2 are points on the boundaries of H_1 and H_2, respectively.

If P_1 and P_2 are vertices of H_1 and H_2, then $P_{\frac{1}{2}}$ is a vertex of $H_{\frac{1}{2}}$. The number of such vertices is finite because the number of combinations of vertices of H_1 and H_2 is finite. If P_1 and P_2 lie on parallel edges of H_1 and H_2, or if one of these points lies on an edge and the other is a vertex, then $P_{\frac{1}{2}}$ lies on an edge of $H_{\frac{1}{2}}$. The number of such edges is finite because the number of combinations of parallel edges of H_1 and H_2, and of edges with vertices is finite. Finally, if one (or both) of the points P_1 or P_2 lies inside a face of H_1 or H_2, or if both points lie in nonparallel edges, then point $P_{\frac{1}{2}}$ belongs to a face of $H_{\frac{1}{2}}$. The number of such faces is also finite.

Thus, having exhausted all possibilities, we see that every point of the boundary of $H_{\frac{1}{2}}$ either is a vertex or belongs to an edge or face, and that the number of vertices, edges, and faces is finite. Therefore $H_{\frac{1}{2}}$ is a polyhedron.

36. PROOF OF ALEKSANDROV'S THEOREM

Proof. Let H_1 and H_2 be two convex polyhedra having corresponding faces with parallel external normals. Thus to each face of one polyhedron there corresponds a face of the other with a parallel external normal and conversely. We are given that of each pair of corresponding faces neither can be imbedded inside the other by a translation.

We shall show that under these conditions polyhedra H_1 and H_2 are congruent by translation, that is, either of them can be translated to coincide with the other. (Note that this translation causes each pair of corresponding faces to coincide, although neither can be imbedded *inside* the other.)

For the proof let us construct the polyhedron

$$H_{\frac{1}{2}} = \frac{1}{2}(H_1 + H_2).$$

Each edge of polyhedron $H_{\frac{1}{2}}$ arises either by mixing a pair of parallel edges of polyhedra H_1 and H_2 lying in supporting planes with parallel external normals or by mixing an edge of one of these polyhedra with a vertex of the other, where the edge and vertex lie in supporting planes with parallel external normals. We may also consider the latter case to be the mixing of two parallel edges one of which has length zero. Under this convention we may say that each edge of polyhedron $H_{\frac{1}{2}}$ corresponds to a pair of edges of polyhedra H_1 and H_2, and arises as the result of mixing them. This convention concerning edges with length zero is the same as that used in section 34.

Now let us mark each edge of $H_{\frac{1}{2}}$ with a plus or minus sign according to whether the associated edge of H_1 is longer or shorter than the corresponding edge of H_2. If they are equal, we make no mark. (Remember the convention about edges of length zero.) We shall show that in making a circuit around any face of polyhedron $H_{\frac{1}{2}}$, if even one edge is marked, then we must encounter at least four changes of sign.

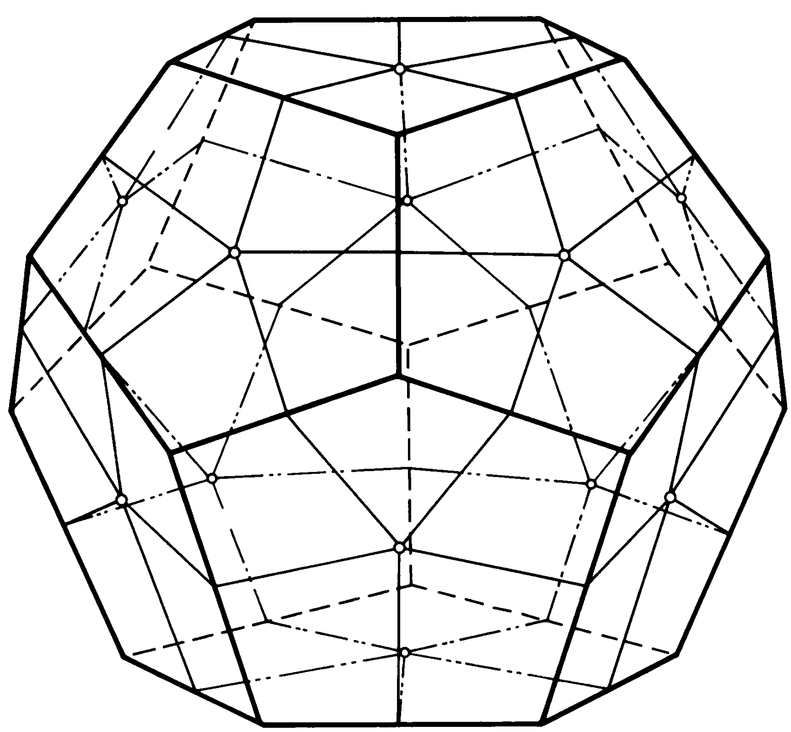
We may divide the faces of polyhedron $H_{\frac{1}{2}}$ into two groups. The first group consists of faces obtained by mixing a pair of faces of H_1 and H_2 with parallel external normals. The second group consists of faces obtained by mixing a pair of nonparallel edges lying in supporting planes with parallel external normals. (In general, as stated in Theorem 5, section 35, a face may result from the mixture of a face with an edge or a vertex, but this cannot happen here because of the conditions imposed upon the correspondence between the faces of polyhedra H_1 and H_2.)

We consider faces of the first group. Since the conditions on corresponding faces of polyhedra H_1 and H_2 do not allow one face to be imbedded inside the other by a translation, then for each pair of corresponding faces we have two possibilities. Either the two faces are congruent by translation, or they are not. If they are, then all the edges of either face are equal to the edges of the other, and none of the edges of the corresponding face of $H_{\frac{1}{2}}$ will be marked. In the second case, by Theorem 2, section 34, we must have at least four changes of sign. (In going around either of these faces, the edges of H_1 become now shorter, now longer than the corresponding edges of the other face at least four times. Hence, there must be at least four changes of sign in going around the corresponding face of $H_{\frac{1}{2}}$.) Thus our assertion is correct for the first group of faces.

Each face $P_{\frac{1}{2}}$ in the second group, being the result of mixing two nonparallel edges, is a parallelogram. Let L_1 be the edge of H_1 and L_2 the edge of H_2 which give $P_{\frac{1}{2}}$ when mixed. Referring to the exposition in section 35 of the mixing of two nonparallel segments, we see that edge L_1, when mixed with the two end points of edge L_2, gives the two opposite sides of parallelogram $P_{\frac{1}{2}}$ parallel to L_1. Here edge L_1 of H_1 is mixed with an edge of H_2 with length zero. We have an analogous situation concerning L_2. Thus we see that we encounter exactly four changes of sign in going around $P_{\frac{1}{2}}$ (or in going around any face in the second group). So our assertion is also true for the second group of faces.

Now we are faced with two possibilities: First, it could be that all corresponding faces of the polyhedra H_1 and H_2 are congruent by translation, so that no edges of polyhedron $H_{\frac{1}{2}}$ are marked by signs. (In this case polyhedron $H_{\frac{1}{2}}$ has no faces of the second group, since if L_1 and L_2 are edges of H_1 and H_2, then they belong to some faces; but corresponding faces are congruent by translation, so their edges are parallel.) If this is so, then polyhedra H_1 and H_2 are congruent by translation. Second, it could be that not all corresponding faces of polyhedra H_1 and H_2 are congruent by translation, so that polyhedron $H_{\frac{1}{2}}$ will have edges marked with signs. In going around a face with marked edges we always encounter at least four changes of sign. We shall show that this second "possibility" is really impossible, which will finish the proof of our theorem.

Fig. 139



For the proof let us choose one point inside each face of the polyhedron $H_{\frac{1}{2}}$ and connect these points by lines crossing the edges along which adjacent faces meet (Fig. 139). We obtain a network consisting of nodes and lines. The nodes (points we chose) correspond to the faces of the polyhedron, and the lines correspond to the edges (the edge the line crosses). The lines converging to one node correspond to the edges of the face in which the node lies. If an edge is marked by a plus or minus, then so shall be the corresponding line. Going around a face corresponds to going around a node. In going around a node with marked lines we encounter at least four changes of sign.

Now we have the same situation as in the proof of Cauchy's theorem. True, there we discussed edges and vertices, and here we discuss lines and nodes. But this difference is not significant, since the derivation of Euler's theorem, which was used in proving Cauchy's theorem, did not require that the lines in the network be straight. We must only require that the network be drawn on a convex surface and that each region be bounded by at least three lines.

In our case these requirements are satisfied. A region corresponds to a vertex of polyhedron $H_{\frac{1}{2}}$, and at least three edges meet in each vertex, and so each of our regions is bounded by at least three lines. It would be sufficient for us to repeat the same arguments with which we proved Cauchy's theorem (cf section 21). Then we would see that it is impossible for polyhedron $H_{\frac{1}{2}}$ to have any marked edges. This completes the proof.

6. Supplement

37. PRECISE DEFINITION OF A CONVEX FIGURE

We shall now give precise definitions of some concepts we have used in the preceding chapters. We shall be considering sets of points or, briefly, "sets." A circle, a polygon, and a lattice of integers are all examples of sets.

DEFINITION. *Let ϵ be some positive number. An* ϵ-neighborhood *of a point A on a line (or in a plane or in space) is defined as the set of all points of the line (or plane or space) having distance from A less than ϵ.*

On the line, this neighborhood is the interior of a segment of length 2ϵ with center at point A. In a plane it is the interior of a circle with radius ϵ and center at A, and in space it is the interior of a sphere with radius ϵ and center at A.

DEFINITION. *A point A is called a* limit point *of a set M if every ϵ-neighborhood of A contains points of M other than A.*

For example, the point 1 on a number line is a limit point of the set of points of the form $1 - \dfrac{1}{n}$ $(n = 1, 2, 3, \ldots)$; any point on a circle is a limit point of the set of points inside the circle, and also of the set of points outside the circle.

DEFINITION. *A set A on a line (plane, space) is said to be* bounded *if it is contained in some segment (circle, sphere).*

A bounded set may have an infinite number of elements, as, for example, the set of points of the form $1 - \dfrac{1}{n}$ $(n = 1, 2, 3, \ldots)$ on a line.

We give the following theorem without proof.

BOLZANO-WEIERSTRASS THEOREM. *Suppose that M is a bounded, infinite set in a line, plane, or space. Then there is at least one point in the line, plane, or space that is a limit point of M.*

DEFINITION. *A set is called* closed *if it contains all its limit points.*

For example, a circle including its interior is a closed set; the interior of a circle without the boundary is not a closed set.

DEFINITION. *A closed set that cannot be divided into two closed sets without points in common is called a* continuum.

For example, a line segment is a continuum; a pair of nonintersecting segments form a closed set which is not a continuum.

The word "figure," as we used it in the preceding chapters, usually denoted a continuum.

DEFINITION. *A set is said to be* convex *if together with any two of its points A and B it contains the entire segment AB.*

A segment connecting two points of a set is sometimes called a *chord* of the set; thus, a set is convex if it contains all of its chords.

DEFINITION. *We shall call a closed convex set a* convex figure.

Every convex figure is a continuum.

DEFINITION. *A point A of a set M is called an* interior point *if some ϵ-neighborhood of A is entirely contained in M.*

It follows from Corollary 1 of Theorem 2, section 1, that *the set of interior points of a convex figure is convex.*

The concept of n-dimensional Euclidean coordinate space is useful in many mathematical problems and constructions. A point of such a space is a set of n numbers $(x_1, x_2, \ldots, x_n)$, which are the coordinates of the point x. We shall write $x(x_1, x_2, \ldots, x_n)$.

DEFINITION. *The* distance $\rho(x, y)$ *between points* $x(x_1, x_2, \ldots, x_n)$ *and* $y(y_1, y_2, \ldots, y_n)$ *is defined by the formula*

$$\rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

DEFINITION. *The* segment *connecting points* $x(x_1, x_2, \ldots, x_n)$ *and* $y(y_1, y_2, \ldots, y_n)$ *is the set of all points of the form*

$$(sx_1 + s_1y_1, sx_2 + s_1y_2, \ldots, sx_n + s_1y_n),$$

where $0 \leq s \leq 1$ and $s + s_1 = 1$.

The concepts of ϵ-neighborhood, closed set, convex set, and so forth are defined in the same way for n-dimensional spaces as for

1-, 2-, and 3-dimensional spaces. Many theorems proved for convex solids can immediately be generalized to the n-dimensional case. But some problems present great difficulties in the n-dimensional case. An example is the construction of the theory of n-dimensional parallelohedra.

38. CONTINUOUS MAPPING AND FUNCTIONS

DEFINITION. *Given sets M and M_1 with a correspondence between them such that to each point, a, of M there corresponds exactly one point, b, of M_1.*

Point b is called the image *of point a.*

The correspondence is called a mapping *or a* transformation of M into M_1.

A function f *is the set of ordered pairs* (a, b); *we write* $b = f(a)$.

DEFINITION. *If each point of set M_1 is the image of one and only one point of set M, then the correspondence between the points of set M and those of set M_1 is called a* one-to-one mapping of M onto M_1.

DEFINITION. *A mapping of a set M into a set M_1 is called a* continuous mapping *if every sequence $a_1, a_2, \ldots, a_n, \ldots$ of points of M having a limit point a in M is mapped onto a sequence $b_1, b_2, \ldots, b_n, \ldots$ of points of M_1 having a limit point b in M_1, where b is the image of point a.*

Projection on an axis, translations, and symmetric transformations are examples of continuous mappings.

DEFINITION. *A* homeomorphism, *or* topological transformation, *of one set M onto another set M_1, is a one-to-one correspondence between them which is continuous both ways.* (*It establishes continuous mappings of M onto M_1 and of M_1 onto M.*)

Translations, reflections, and isometric transformations are examples of homeomorphisms. The central projection described in section 9 is a homeomorphism between two convex surfaces.

Topology is that part of geometry which deals with those properties of sets that are preserved under homeomorphisms. Euler's theorem belongs to topology because it concerns networks on any surface that is topologically equivalent (homeomorphic) to the surface of a sphere.

The concept of convex set can be used in a more general sense. Suppose that we are given a set C of *continuous functions* y defined on the interval $0 \leq x \leq 1$.

DEFINITION. *For two continuous functions in C, y_0 and y_1, the system of functions defined by*

$$y_s(x) = sy_0(x) + s_1y_1(x),$$

where $0 \leq s \leq 1$, $s + s_1 = 1$, *is called the* segment connecting the functions *y_0 and y_1 of C.*

DEFINITION. *A set C of functions is called a* convex set of functions *if it contains the segment connecting any two functions in it.*

Two examples of convex sets of functions are the set of all polynomials, and the set of all functions y satisfying the inequality $|y(x)| \leq 1$ for all x.

The extension of the properties of ordinary convex figures to such more general sets is important in mathematical analysis.

39. REGULAR NETWORKS; REGULAR AND SEMIREGULAR POLYHEDRA

DEFINITION. *A connected network is called a* regular network *if each of its regions is bounded by the same number of lines and the same number of lines meet at each node.*

The following are regular networks:

(*a*) The network formed by the vertices, edges, and faces of a regular convex polyhedron, that is, a *tetrahedron, cube, octahedron, icosahedron,* or a *dodecahedron.*

(*b*) The network formed by one l-gon dividing a surface into two parts. Two lines of the network meet in each node; each of the two regions is bounded by l lines (sides).

(*c*) A network consisting of two nodes connected by n lines. At each node meet n lines; each of the n regions is bounded by two lines.

We shall prove the following:

THEOREM 1. *There are no other regular networks than those listed on page* 153.

Proof. Let p denote the number of lines meeting in each node, and q the number of lines bounding each region; l, m, and n denote, as before, the numbers of lines, nodes, and regions of the network. We have

$$l = \frac{1}{2}mp = \frac{1}{2}nq,$$

because each of the n regions is bounded by q lines and each line belongs to two regions; p lines meet in each of the m nodes and each line connects two nodes. Hence

$$m = \frac{2l}{p}, \qquad n = \frac{2l}{q}. \tag{1}$$

We can write Euler's equation $m + n - l = 2$ in the form

$$l\left(\frac{2}{p} + \frac{2}{q} - 1\right) = 2$$

or

$$\frac{2}{p} + \frac{2}{q} - 1 = \frac{2}{l}. \tag{2}$$

For $q = 2$ we obtain case (c); for $p = 2$ we get case (b).

Now assume $p \geq 3$, $q \geq 3$. It is easy to prove that this implies $p \leq 5$. If $p \geq 6$ and $q \geq 3$ we have

$$\frac{2}{p} + \frac{2}{q} - 1 \leq \frac{1}{3} + \frac{2}{3} - 1 = 0,$$

which contradicts equation (2). Analogously it is shown that $q \leq 5$.

If $p = 4$ or $p = 5$, then $q = 3$. For the proof, suppose $p \geq 4$ and $q \geq 4$. Then

$$\frac{2}{p} + \frac{2}{q} - 1 \leq \frac{1}{2} + \frac{1}{2} - 1 = 0,$$

which contradicts (2). Analogously, if $q = 4$ or $q = 5$, then $p = 3$.

Thus there are five possible cases:

(i) $p = 3$, $q = 3$. From equation (2) we get

$$\frac{2}{l} = \frac{2}{3} + \frac{2}{3} - 1 = \frac{1}{3}, \qquad l = 6.$$

From (1):

$$m = \frac{2l}{p} = \frac{12}{3} = 4; \qquad n = \frac{2l}{q} = \frac{12}{3} = 4.$$

This gives a *tetrahedron.*

(ii) $p = 4$, $q = 3$. From (2):

$$\frac{2}{l} = \frac{2}{4} + \frac{2}{3} - 1 = \frac{1}{6}, \qquad l = 12.$$

From (1):

$$m = \frac{2l}{p} = \frac{24}{4} = 6; \qquad n = \frac{2l}{q} = \frac{24}{3} = 8.$$

This gives an *octahedron.*

(iii) $q = 4, p = 3$. Analogously, we obtain $l = 12, m = 8, n = 6$. This gives a *cube.*

(iv) $p = 5$, $q = 3$. From (2):

$$\frac{2}{l} = \frac{2}{5} + \frac{2}{3} - 1 = \frac{1}{15}, \qquad l = 30.$$

From (1):

$$m = \frac{2l}{p} = \frac{60}{5} = 12; \qquad n = \frac{2l}{q} = \frac{60}{3} = 20.$$

This gives an *icosahedron.*

(v) $q = 5, p = 3$. Analogously, we obtain $l = 30, m = 20, n = 12$. This gives a *dodecahedron.*

No other regular networks exist. This completes case (a) and the proof the theorem.

In addition to regular *convex* polygons, there exist regular *star-shaped* polygons. We give the following examples: a regular star-shaped pentagon (Fig. 140), a star-shaped hexagon composed of two triangles (Fig. 141), two types of regular star-shaped heptagons (Fig. 142, 143), star-shaped octagon (Fig. 144) and an octagon composed of two squares (Fig. 145).

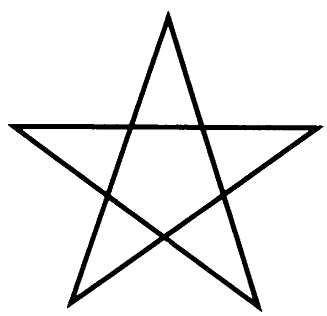


Fig. 140

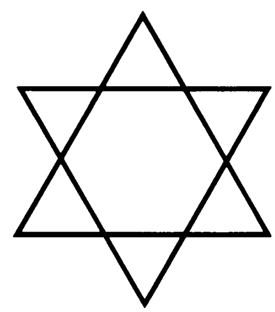


Fig. 141

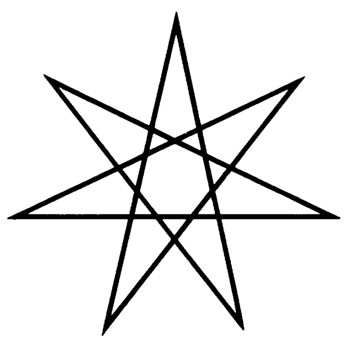


Fig. 142

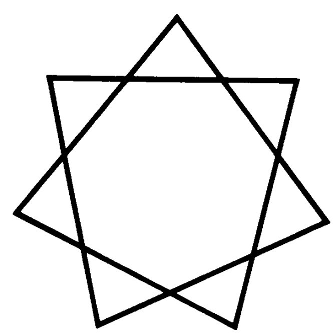


Fig. 143

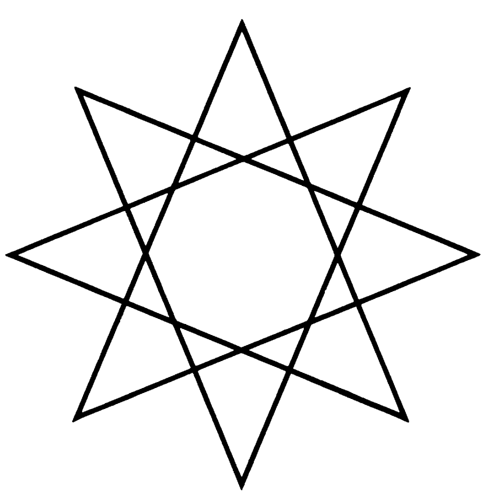


Fig. 144

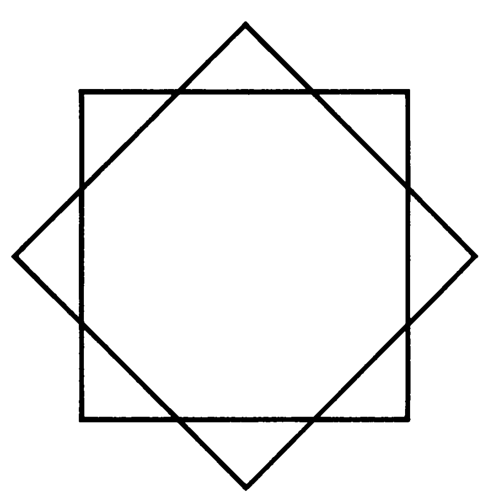
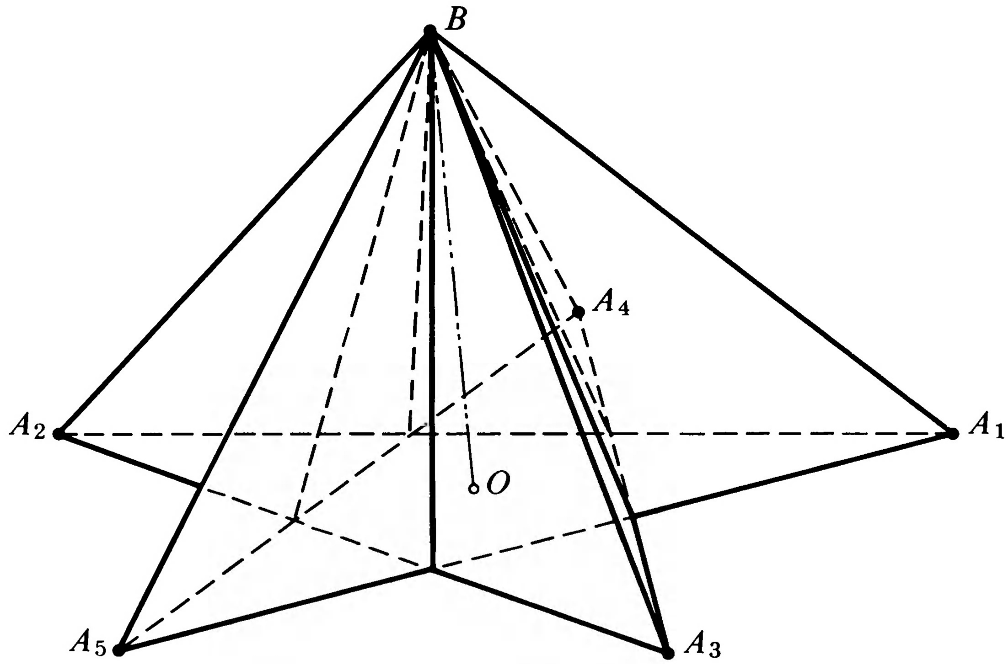


Fig. 145

Fig. 146

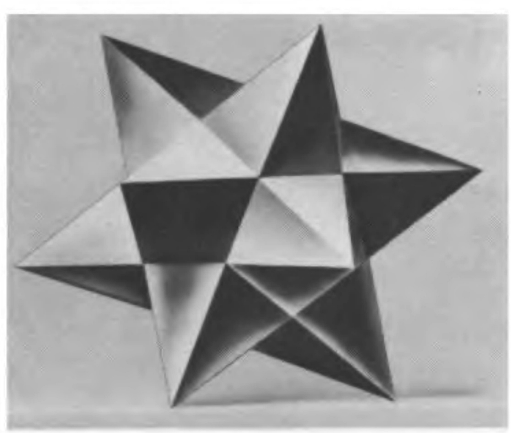


In addition to regular *convex* polyhedral angles, there exist regular *star-shaped* polyhedral angles. In Figure 146 we illustrate a regular star-shaped pentahedral angle. If we erect a perpendicular OB at the center O of a regular star-shaped pentagon $A_1A_2A_3A_4A_5$, then the five equal plane angles A_1BA_2, A_2BA_3, A_3BA_4, A_4BA_5, A_5BA_1 form a regular star-shaped pentahedral angle.

The five regular convex polyhedra, known as the *Platonic* solids, were known in the ancient world. In the 17th century Kepler (1571–1630) discovered the existence of two *star-shaped* or *stellated regular polyhedra.* In 1810 L. Poinsot (1777–1859) discovered the existence of four such polyhedra, which were given the name *regular generalized polyhedra.* They are:

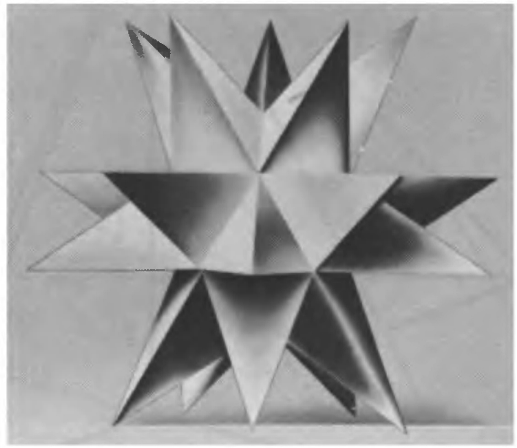
(1) *Small star-shaped* or *stellated dodecahedron* (Fig. 147). It has 12 faces, appearing as star-shaped pentagons, 30 edges, and 12 vertices, appearing as the vertices of regular convex pentahedral angles.

Fig. 147



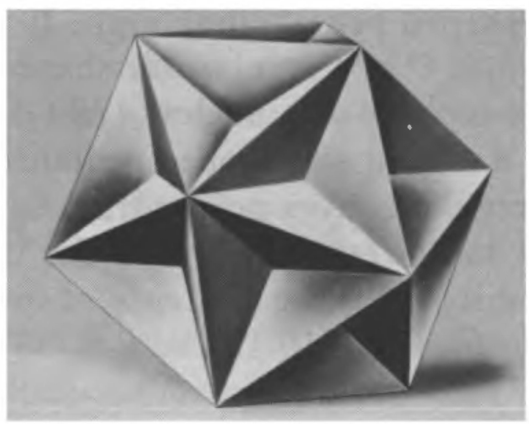
(2) *Great star-shaped* or *stellated dodecahedron* (Fig. 148). It has 12 faces, appearing as regular star-shaped pentagons, 30 edges, and 20 vertices, appearing on regular trihedral angles.

Fig. 148



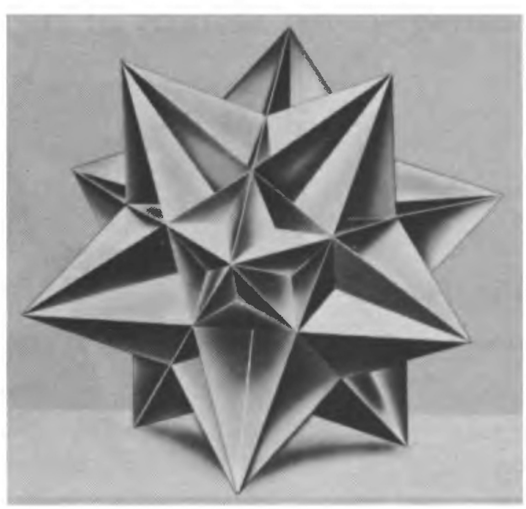
(3) *Great dodecahedron* (Fig. 149) with 12 faces, appearing as regular convex pentagons, 30 edges, and 12 vertices, appearing on regular star-shaped pentahedral angles.

Fig. 149



(4) *Great icosahedron* (Fig. 150) with 20 faces, appearing as equilateral triangles, 30 edges, and 12 vertices, appearing on regular star-shaped pentahedral angles.

Fig. 150



In 1812 Cauchy proved that the only possible regular polyhedra are the five Platonic solids, the four regular generalized polyhedra, and the octahedron composed of two tetrahedra (Fig. 151).

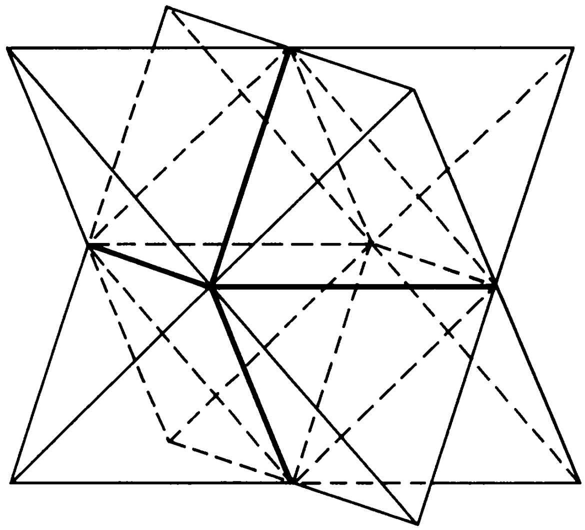


Fig. 151

In 1880 the American mathematician Stringham investigated regular convex polyhedra in n-dimensional space. We shall let d_0 denote the number of their vertices, d_1, the number of edges, d_2, the number of plane (two-dimensional) faces, d_3, the number of three-dimensional faces, and so on. In four-dimensional space there turn out to be six regular polyhedra:

(1) Four-dimensional *5-cell* (the analog of a tetrahedron): $d_0 = 5$, $d_1 = d_2 = 10$, $d_3 = 5$ (5 tetrahedra as three-dimensional faces).

(2) Four-dimensional *8-cell* or *tessaract:* $d_0 = 16$, $d_1 = 32$, $d_2 = 24$, $d_3 = 8$ (8 cubes as three-dimensional faces).

(3) Four-dimensional *16-cell* (the analog of an octahedron): $d_0 = 8$, $d_1 = 24$, $d_2 = 32$, $d_3 = 16$ (16 three-dimensional tetrahedra).

(4) Four-dimensional *24-cell:* $d_0 = 24$, $d_1 = d_2 = 96$, $d_3 = 24$ (24 three-dimensional octahedra).

(5) Four-dimensional *120-cell:* $d_0 = 600$, $d_1 = 1{,}200$, $d_2 = 720$, $d_3 = 120$ (120 three-dimensional dodecahedra).

(6) Four-dimensional *600-cell:* $d_0 = 120$, $d_1 = 720$, $d_2 = 1{,}200$, $d_3 = 600$ (600 tetrahedra).

In n-dimensional space for $n \geq 5$, there exist only three types of regular convex polyhedra. They are the analogs of the tetrahedron, cube, and octahedron.

Remark. The numbers d_0, d_1, d_2, d_3, for any four-dimensional convex polyhedron (in particular, for any regular polyhedron) are related by the equation

$$d_0 - d_1 + d_2 - d_3 = 0.$$

Remember that for $n = 3$ we have

$$d_0 - d_1 + d_2 = 2$$

by Euler's theorem. For $n = 2$ a convex polyhedron reduces to a convex polygon, for which the number d_0 of vertices is equal to the number d_1 of sides (edges), that is,

$$d_0 - d_1 = 0.$$

For $n = 1$ a convex polyhedron reduces to a straight line segment, for which

$$d_0 = 2$$

(the two vertices at the ends of the segment).

In general, for an n-dimensional convex polyhedron the numbers d_k ($k = 0, 1, 2, \ldots, n - 1$) are related by the equation

$$d_0 - d_1 + d_2 - \cdots + (-1)^{n-1}d_{n-1} = 1 + (-1)^{n-1}$$

(the Euler-Poincaré formula). The right side of this formula is equal to 0 for n even (for example, $n = 2, 4$) and equal to 2 for n odd (for example, $n = 1, 3$).

This formula, generalizing the formula of Euler (for the case $n = 3$), is itself a special case of the general Euler-Poincaré formula giving the value of such a signed sum (the Euler characteristic) for any polyhedron with any number of dimensions.

DEFINITION. *A polyhedron is called a* semiregular polyhedron *if all its faces are regular polygons (not necessarily equal) and all its polyhedral angles are equal.*

The simplest examples of semiregular polyhedra are the semiregular *prisms,* having regular n-gons ($n = 3, 4, 5, \ldots$) as bases and squares as lateral faces, and the semiregular *prismoids,* having regular n-gons ($n = 3, 4, 5, \ldots$) as bases and $2n$ equilateral triangles as lateral faces (Fig. 152). Archimedes showed that besides these two series, there exist 13 types of semiregular polyhedra (called the Archimedean solids). In the third century A.D. Pappus wrote an ex-

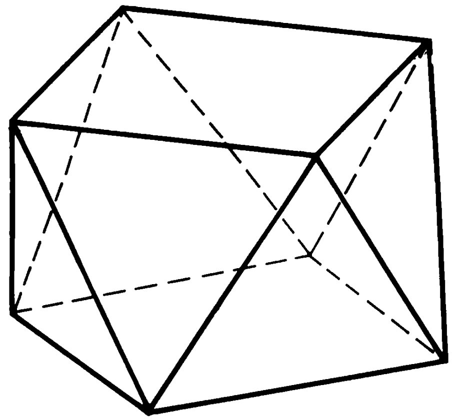


Fig. 152

planation of the work of Archimedes. In Book II of *Harmonices Mundi* Kepler established the complete theory of semiregular polyhedra.

We shall give drawings and the names of these 13 semiregular polyhedra.

(1) *Snub cube* (Fig. 153) has 32 triangular and 6 square faces.

(2) *Cuboctahedron* (Fig. 154) having 8 triangular and 6 square faces.

(3) *Small rhombicuboctahedron* (Fig. 155) formed by 8 triangles and 18 squares.

(4) *Snub dodecahedron* (Fig. 156) having 80 triangles and 12 pentagons as faces.

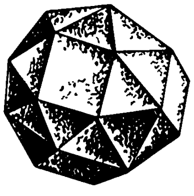


Fig. 153

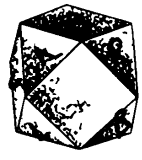


Fig. 154

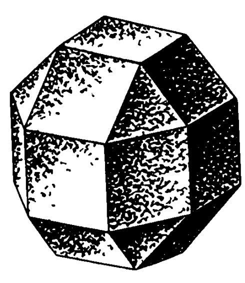


Fig. 155

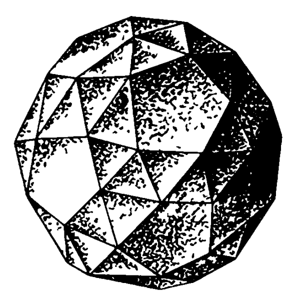


Fig. 156

(5) *Icosidodecahedron* (Fig. 157) formed by 20 triangles and 12 pentagons.

(6) *Truncated tetrahedron* (Fig. 158) has 4 triangles and 4 hexagons as faces.

(7) *Truncated cube* (Fig. 159) consisting of 8 triangles and 6 octagons.

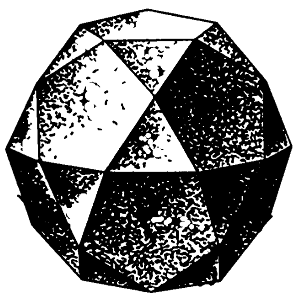


Fig. 157

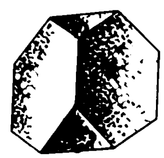


Fig. 158

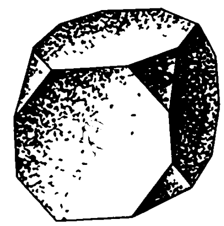


Fig. 159

(8) *Truncated dodecahedron* (Fig. 160) with 20 triangles and 12 decagons as faces.

(9) *Truncated octahedron* (Fig. 161) formed by 6 squares and 8 hexagons.

(10) *Truncated icosahedron* (Fig. 162) consisting of 12 pentagons and 20 hexagons.

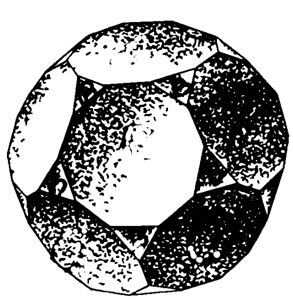


Fig. 160

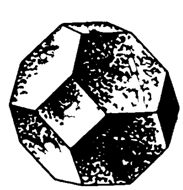


Fig. 161

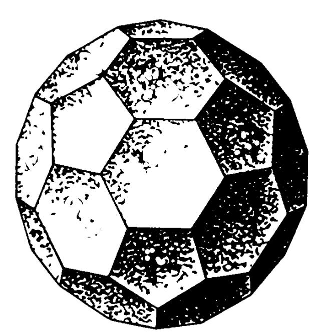


Fig. 162

(11) *Small rhombicosidodecahedron* (Fig. 163) formed by 20 triangles, 30 squares, and 12 pentagons.

(12) *Great rhombicuboctahedron* (Fig. 164) consisting of 12 squares, 8 hexagons, and 6 octagons.

(13) *Great rhombicosidodecahedron* (Fig. 165) having 30 squares, 20 hexagons, and 12 decagons as faces.

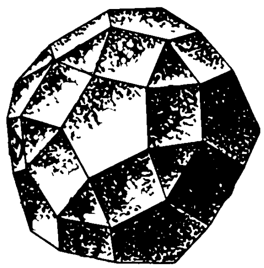


Fig. 163

Fig. 164

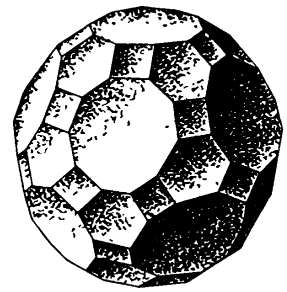


Fig. 165

It is remarkable that in the theory of semiregular polyhedra, which is over 2,000 years old, there was a defect, which was recently discovered by the Soviet mathematician V. G. Ashkinuz. He discovered a 14th semiregular polyhedron (Fig. 166), which differs

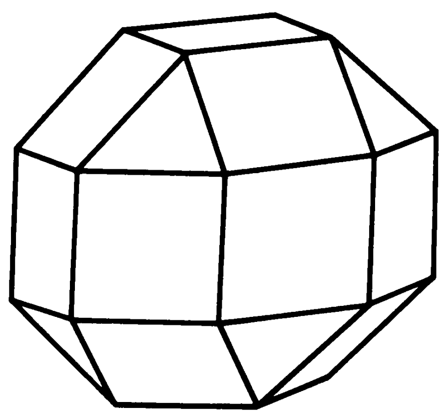


Fig. 166

from the one shown in Figure 155 only in that the upper part, consisting of 5 squares and 4 equilateral triangles, is rotated through an angle of $\pi/4$. In the past these two seimregular polyhedra were not distinguished.

There also exist semiregular star polyhedra. At the present time 51 such polyhedra are known, but no one has proved that they exhaust all such polyhedra.

40. THE ISOPERIMETRIC PROBLEM

In section 32 we proved that of all plane convex figures with given perimeter the circle has the greatest area. This property is expressed by the inequality $l^2 \geq 4\pi S$, where l and S are the perimeter and area of convex figure Q, with equality holding if and only if Q is a circle. Below we shall prove:

THEOREM 2. *The circle has the greatest area among all figures (convex or not) with a given perimeter. In particular, the perimeter l and area S of a nonconvex figure satisfy the strict inequality*

$$l^2 > 4\pi S. \tag{1}$$

We shall base the proof on the following.

DEFINITION. *By the* convex hull *of a set Q we mean the smallest convex figure Q_0 containing Q.*

LEMMA. *Let Q be an arbitrary figure (continuum) and Q_0 be its convex hull* (Fig. 167). *Let q denote the boundary of Q. Then the boundary q_0 of convex figure Q_0 consists of points of q and of straight line segments such as AB whose end points A and B belong to q.*

Proof. It is easy to see that q_0 consists of points belonging to q and of arcs, $\overset{\frown}{AKB}$, connecting them, where the end points A and B of each such arc belong to q but the remaining points do not belong to q and hence do not belong to Q. We shall show that each such arc, $\overset{\frown}{AKB}$, coincides with the straight line segment AB.

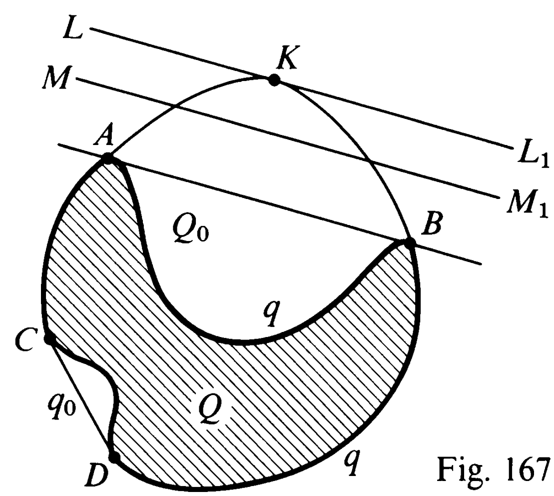


Fig. 167

To prove this, suppose that $\overset{\frown}{AKB}$ does not coincide with AB. Then by Corollary 2 to Theorem 2, section 1, segment AB lies entirely (with the exception of its end points) inside Q_0. One of the two supporting lines for Q_0 parallel to segment AB, the line LL_1, touches $\overset{\frown}{AKB}$ of the boundary of Q_0. But since this arc lies entirely (with the exception of its end points) outside Q, then line LL_1 has no points in common with Q.

It is possible to draw a line MM_1 parallel to LL_1 and lying between it and AB so that MM_1 will have no points in common with Q. Removing from Q_0 the part contained between MM_1 and LL_1, we obtain a convex figure Q_1 smaller than Q_0 but also containing Q. Thus Q_0 cannot be the smallest convex figure containing Q, in contradiction to our assumption. This contradiction proves that $\overset{\frown}{AKB}$ coincides with segment AB.

Proof of Theorem 2. Let l and S denote the perimeter and area of Q, and l_0 and S_0 denote the perimeter and area of Q_0. If Q is not a convex figure, then Q is strictly smaller than its convex hull Q_0:

$$S < S_0. \tag{2}$$

Also, $l > l_0$. To prove this, let CD be a straight line segment (from the Lemma) belonging to the boundary q_0 of the hull Q_0 where C and D are points of q, $\overset{\frown}{CD}$ is an arc forming part of the boundary of Q with end points C and D, and $\overset{\frown}{CD}$ is different from CD. Then the length of $\overset{\frown}{CD}$ is greater than the length of CD (Fig. 167). In passing from Q to Q_0, $\overset{\frown}{CD}$ of the boundary q is replaced by the shorter segment CD of the boundary q_0. Hence the length of the boundary decreases in passing from Q to Q_0, that is,

$$l > l_0. \tag{3}$$

Since Q_0 is a convex figure, its perimeter l_0 and area S_0 satisfy the inequality (section 32)

$$l_0^2 \geq 4\pi S_0.$$

From this and inequalities (2) and (3), we have

$$l^2 > 4\pi S,$$

which we were to prove.

We may remark that the analogous theorem holds for the three-dimensional case:

Among all solids with given surface area the sphere has the greatest volume, and, conversely, among all solids with given volume the sphere has the smallest surface area.

41. CHORDS OF ARBITRARY CONTINUA; LEVI'S THEOREM

Earlier we noted that a line segment connecting two points of a set Q is called a chord of Q and that a convex figure contains all of its chords. However, a continuum that is not a convex figure does not have this property, and it appears difficult to say anything definite about the chords of an arbitrary plane continuum. The following theorem becomes all the more interesting.

LEVI'S THEOREM. *If a bounded plane continuum has a chord of length a, then it has chords parallel to it with lengths* $\frac{a}{n}$ $(n = 1, 2, 3, \ldots)$. *But if* α *is a number different from all the numbers* $\frac{1}{n}$ $(n = 1, 2, 3, \ldots)$, *then there exists a bounded continuum having a chord of length a, but not having a chord parallel to it with length* αa.

Thus the numbers which are the inverses of the natural numbers appear in a privileged position.

Proof. We shall present an outline of Heinz Hopf's proof of Levi's theorem. (All continua mentioned are assumed to be bounded.) Let S be a plane continuum. We shall consider those of its chords that are parallel to some line that we choose as the x-axis. Let S_a denote the continuum obtained by translating S by a vector of length a parallel to the x-axis.

It is easy to show that the continua S and S_a intersect or do not intersect according to whether S has or does not have a chord of length a parallel to the x-axis. For the proof, suppose first that S has a chord AB with length a parallel to the x-axis (Fig. 168). Under the translation parallel to the x-axis by a vector of length a, S goes

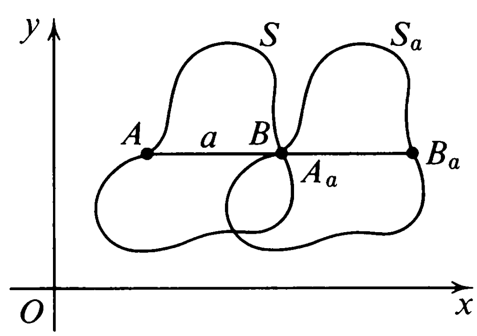


Fig. 168

into S_a, and point A of continuum S goes into point B, which therefore belongs also to continuum S_a. Hence B is an intersection point of S and S_a.

Now suppose that S does not have a single chord of length a parallel to the x-axis. Each point A of continuum S corresponds to some point A_1 of continuum S_a such that segment AA_1 is parallel to the x-axis and has length equal to a. If S and S_a had a point A_1 in common, then together with the corresponding point A of continuum S it would define a chord AA_1 of S with length a parallel to the x-axis. But we assumed that no such chord exists. Hence in this case S does not intersect S_a.

LEMMA. *If S has no chords parallel to the x-axis of length a or b, then S has no chord parallel to the x-axis of length $a + b$.*

Proof. By virtue of the above remarks, it is sufficient to prove that if S does not intersect either S_a or S_b, then S does not intersect S_{a+b}.

Notice that if we translate both S and S_a by a vector parallel to the x-axis having length b, then they become S_b and S_{a+b}. Since S and S_a do not intersect, S_b and S_{a+b} do not intersect (Fig. 169).

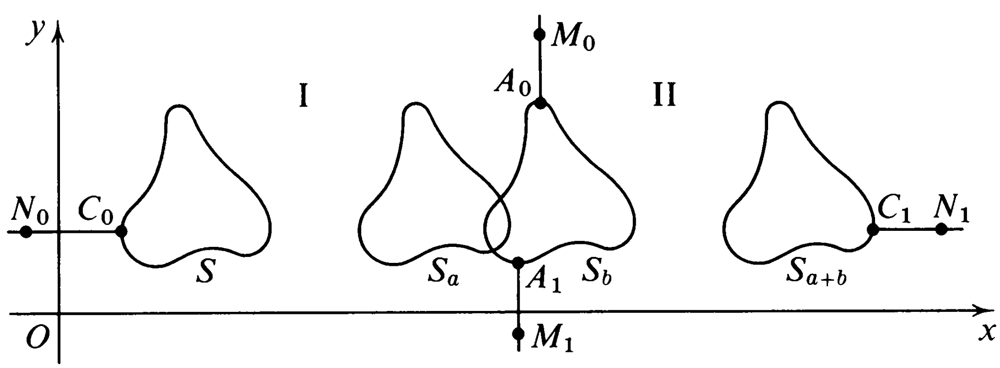


Fig. 169

Let A_0 and A_1 denote points of S_b having the greatest and least possible ordinates. Draw two infinite half-lines parallel to the y-axis: From point A_0 draw the half-line A_0M_0 in the direction of increasing y; from point A_1 draw the half-line A_1M_1 in the direction of decreasing y. Half-lines A_0M_0 and A_1M_1 intersect neither of the continua S and S_{a+b}. We prove this as follows: Since point A_0 has the greatest ordinate of the points of S_b, it has the greatest ordinate of points of S and S_{a+b} as well. All other points of half-line A_0M_0 have ordinates greater than those of any points of S and S_{a+b}. Hence A_0M_0 intersects neither S nor S_{a+b}. We prove analogously that A_1M_1 intersects neither S nor S_{a+b}.

Continuum S_b together with half-lines A_0M_0 and A_1M_1 forms an infinite figure T which divides the plane into two parts, I and II, where part I contains the points with the smaller abscissas, and part II the larger. With this division S lies in part I and S_{a+b} in part II.

Let C_0 be a point of S with the smallest possible abscissa (a leftmost point of S). Its abscissa must be smaller than the abscissas of all the points of S_b, because S_b is obtained from S by moving it to the right (the direction of increasing abscissas). Therefore the half-line C_0N_0 parallel to the x-axis drawn from C_0 in the direction of decreasing abscissas does not intersect S_b. Both S and this half-line lie entirely in part I of the plane.

Analogously, if C_1 is a point of S_{a+b} with the greatest possible abscissa (a rightmost point), then the half-line C_1N_1 drawn from C_1 parallel to the x-axis in the direction of increasing abscissas does not intersect S. Hence, both this half-line and S_{a+b} lie entirely in part II of the plane.

Thus, S and S_{a+b} lie in different parts of the plane and hence have no points in common, and the lemma is proved.

Proof of Levi's theorem (continued). If $b = a$, the lemma gives us the following conditions: If a continuum does not have a chord of length b parallel to the x-axis, then it does not have such a chord of length $2b$; also, since it has no chords parallel to the x-axis of length b or $2b$, then it has no such chord of length $3b$, and so forth. In general, it has no chord parallel to the x-axis with length nb, for any natural number n.

From this follows the first part of the theorem: If a continuum S has no chord of length $\frac{a}{n}$ $(n = 2, 3, 4, \ldots)$ parallel to some line, then it can have no chord of length $n\frac{a}{n} = a$ parallel to the same line. Hence, if S has a chord of length a, then it has chords parallel to it of lengths $\frac{a}{n}$ for all natural numbers $n (n = 2, 3, 4, \ldots)$.

Now we shall prove that if $\alpha \neq \frac{1}{n}$ for all $n = 1, 2, 3, \ldots$, then it is possible to construct a continuum (even a broken line) having a chord of length 1 but not having a chord of length α.

Case I. Let $\alpha > 1$. Consider a line segment of length 1. It has a chord of length 1 but none of length α.

Case II. There remains the case

$$0 < \alpha < 1 \quad (\alpha \neq \frac{1}{2}, \frac{1}{3}, \ldots).$$

There exists a natural number n such that $\frac{1}{n} > \alpha > \frac{1}{n+1}$; that is, $n\alpha < 1 < (n+1)\alpha$. Hence

$$1 = n\alpha + \beta, \quad \text{where} \quad 0 < \beta = 1 - n\alpha < \alpha.$$

We shall use the following notation. If r denotes the point on the x-axis with abscissa r, then $[r, r_1]$ will denote the segment $r \leq x \leq r_1$ of this axis. Let $B_0 = 0$, $D = 1$ (Fig. 170, where $n = 3$). On segment $B_0D = [0, 1]$ mark the points $\alpha, 2\alpha, \ldots, n\alpha$. Then we choose a number $h > 0$ so small that it satisfies the inequalities:

$$\alpha - 2nh > \beta > 0, \tag{1}$$

$$(n + 1)h < \beta. \tag{2}$$

From (2) it follows that

$$n(\alpha + h) < n\alpha + \beta = 1. \tag{3}$$

Hence, all points $\alpha \pm h, 2(\alpha \pm h), \ldots, n(\alpha \pm h)$ lie in $[0, 1]$. Further, for $i = 0, 1, 2, \ldots, n - 1$ we have by (1)

$$(i + 1)(\alpha - h) - i(\alpha + h) = \alpha - (2i + 1)h > \alpha - 2nh > 0. \tag{4}$$

Each point $(i + 1)(\alpha - h)$ lies to the right of point $i(\alpha + h)$, and the segments $[i(\alpha - h), i(\alpha + h)]$ $(i = 1, 2, \ldots, n)$ (in Fig. 170 $n = 3$) do not intersect.

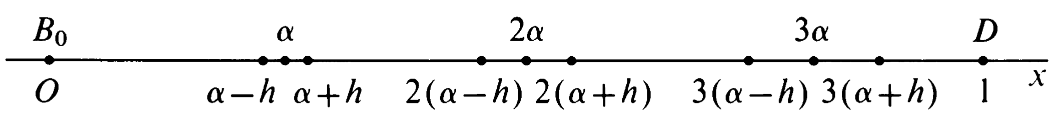


Fig. 170

Now we construct isosceles right triangles having the segments $[i(\alpha - h), i(\alpha + h)]$ of lengths $2ih$ as hypotenuses and the points B_i $(i = 1, 2, \ldots, n)$ (Fig. 171) as vertices, where B_i has abscissa

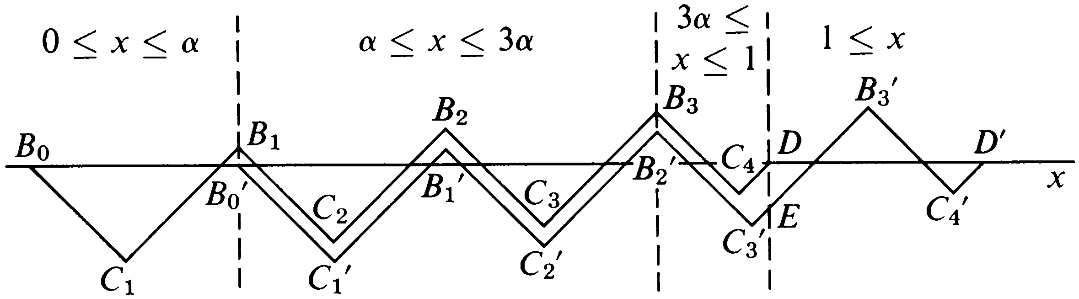


Fig. 171

$i\alpha$ and ordinate ih. The point B_{i+1} is obtained from point B_i by moving the latter to the right by α and upward by h.

Analogously on the segments $[(i - 1)(\alpha + h), i(\alpha - h)]$ $(i = 1, 2, \ldots, n)$ and $[n(\alpha + h), 1]$ we construct isosceles right triangles with vertices $C_1, C_2, \ldots, C_n, C_{n+1}$, where for $i = 1, 2, \ldots,$ n C_i has abscissa

$$\frac{(i - 1)(\alpha + h) + i(\alpha - h)}{2} = -\frac{\alpha + h}{2} + i\alpha$$

and the *negative* ordinate equal in absolute value to half the length of the corresponding segment, that is,

$$-\frac{i(\alpha - h) - (i - 1)(\alpha + h)}{2} = -\frac{\alpha - h}{2} + (i - 1)h.$$

This expression is negative by (3) and (4). Thus each point C_{i+1} $(i = 1, 2, \ldots, n - 1)$ is obtained from the point C_i by moving to the right by α and upward by h.

Finally, C_{n+1} has the negative ordinate equal to half the length of segment $[n(\alpha + h), 1]$, that is, equal to

$$-\frac{1 - n(\alpha + h)}{2},$$

and abscissa

$$\frac{1 + n(\alpha + h)}{2} = \frac{n\alpha + \beta + n\alpha + nh}{2} = n\alpha + \frac{nh + \beta}{2}.$$

Let S denote the broken line $B_0C_1B_1C_2B_2 \ldots B_nC_{n+1}D$ and S_α denote the broken line $B_0'C_1'B_1'C_2'B_2' \ldots B_n'C_{n+1}'D'$ obtained

from S by translation to the right parallel to the x-axis by α. We shall show that S and S_α do not intersect, which proves that S, while it has a chord B_0D of length 1, does not have a chord of length α. After being moved to the right by α, the vertices $C_1, C_2, \ldots, C_{n-1}, B_0, B_1, B_2, \ldots, B_{n-1}$ become $C_1', C_2', \ldots, C_{n-1}', B_0', B_1', B_2', \ldots, B_{n-1}'$ with the same ordinates as before and abscissas increased by α. Each of these new vertices has the same abscissa, and the ordinate decreased by h, as the corresponding vertex $C_2, C_3, \ldots, C_n, B_1, B_2, \ldots, B_n$ of broken line S. Hence, the part $B_1C_2B_2C_3 \ldots B_n$ of broken line S in the region $\alpha \leq x \leq n\alpha$, when moved downward by h, coincides with the part $B_0'C_1'B_1'C_2' \ldots B_{n-1}'$ of broken line S_α in the same region. Hence, the parts of S and S_α lying in this region have no point of intersection.

In the region $x < \alpha$, there are no points of S_α. In the region $x > 1$, there are no points of S. There remains the region $n\alpha < x < 1$ to be examined. The part of S in this region consists of the broken line $B_nC_{n+1}D$. The part of S_α in this region is the broken line $B_{n-1}'C_n'E$ (Fig. 171), the sides of which are parallel to the corresponding sides of broken line $B_nC_{n+1}D$. The point B_{n-1}' has the same abscissa as B_n, but a smaller ordinate. The point C_n has abscissa $-\frac{1}{2}(a + h) + n\alpha$, and the point C_{n+1} has abscissa

$$\frac{1 + n(\alpha + h)}{2} = \frac{n\alpha + \beta + n\alpha + nh}{2} = n\alpha + \frac{nh + \beta}{2}.$$

The abscissa of point C_n' is obtained from the abscissa of point C_n by adding α, so that the abscissa of C_n' is

$$(n + 1)\alpha - \frac{\alpha + h}{2}.$$

If we subtract the abscissa of C_{n+1} from the abscissa of C_n', we obtain

$$(n + 1)\alpha - \frac{\alpha + h}{2} - n\alpha - \frac{nh + \beta}{2} = \frac{\alpha - h - (nh + \beta)}{2}.$$

But by the inequality

$$\alpha - h > \beta + nh,$$

obtained from (1), this difference between the abscissas is positive. Hence, point C_n' has a greater abscissa than C_{n+1}, and therefore the broken lines $B_nC_{n+1}D$ and $B'_{n-1}C_n'E$ do not intersect. Hence, S and S_α do not intersect, and the theorem is proved.

Levi's theorem belongs to that part of geometry which deals with the general metric properties of arbitrary geometric objects. To this subject also belongs the following theorem of the Soviet mathematician L. G. Shnirelman (1905–1938).

For any closed curve q it is possible to construct a square with its vertices on q.

The contents of the next section pertain to this same branch of geometry.

42. FIGURES IN A LATTICE OF INTEGERS; BLICHFELDT'S THEOREM

Consider a lattice of integers in the plane and some two-dimensional figure Q. The number of nodes of the lattice covered by the figure depends on its position. For, if the center of a convex figure with area greater than 4 coincides with one of the nodes, then by the theorem of Minkowski in section 15 the figure also covers other nodes. But there exist central-symmetric convex plane figures with area arbitrarily large or even infinite which do not cover a single node of the lattice (for example, the strip between the parallel lines LL_1, MM_1 in Figure 172).

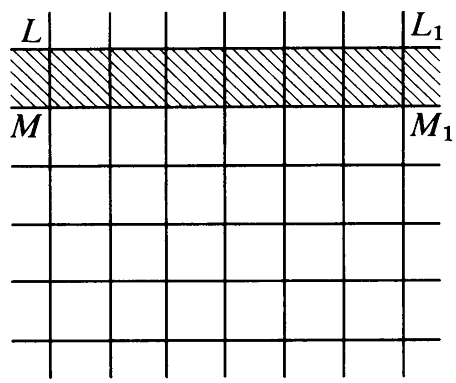


Fig. 172

In connection with this we give an interesting theorem of the American mathematician Blichfeldt.

BLICHFELDT'S THEOREM. *If the area of a two-dimensional figure Q is greater than the whole number n, then Q can be translated by a vector of length less than* 1 *so that it will cover at least $n + 1$ nodes of the lattice.*

From this it follows that if the area of a two-dimensional figure Q is equal to n (hence greater than $n - 1$), then Q can be translated

so that it will cover at least n nodes of the lattice (since $n = (n-1) + 1$).

Proof. A regular integral lattice divides the plane into identical squares and divides the figure Q into parts $A_1, A_2, \ldots, A_k$ lying in individual squares $R_1, R_2, \ldots, R_k$ (in Figure 173 $n = 2$, $k = 4$).

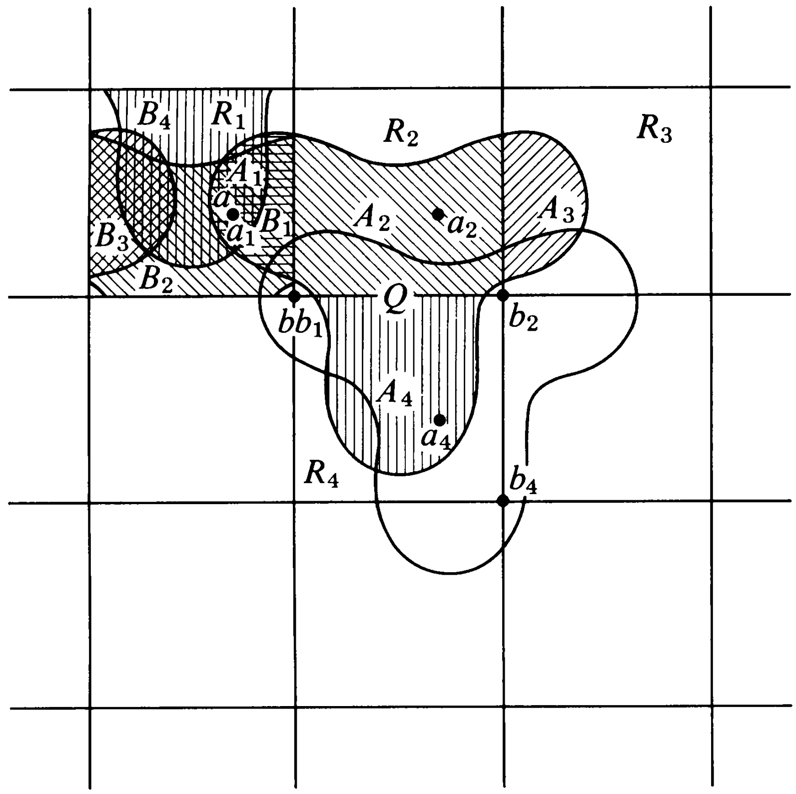


Fig. 173

Choose one of these squares, for example R_1, and translate each of the squares $R_2, R_3, \ldots, R_k$ onto R_1. These translations are given by integral vectors. The parts $A_1, A_2, \ldots, A_k$ of figure Q are carried onto parts $B_1, B_2, \ldots, B_k$ of square R_1 congruent to them. We shall say that a point *a is covered l times* by the parts $B_1, B_2, \ldots, B_k$ if l of these parts cover point a (obviously, $l \leq k$).

LEMMA. *There exists at least one point in square R_1 which is covered at least $n + 1$ times by the parts $B_1, B_2, \ldots, B_k$.*

Proof. Here of course we are assuming that the area of figure Q is greater than n. If each point of square R_1 is covered not more than n times by these parts, then the combined area of these parts cannot be greater than n times the area of R_1, i.e., does not exceed n. This implies that the combined area of all the parts $A_1, A_2, \ldots, A_k$ of figure Q, which are congruent to $B_1, B_2, \ldots, B_k$, is not greater than n. But the area of Q was assumed to be greater than n. This contradiction proves the lemma.

Proof of the theorem (continued). From the lemma it follows that there exists inside R_1 a point a that is covered by at least $n + 1$ of the parts B_i. Each of these parts is obtained by translating the corresponding part A_i by an integral vector. Under this translation some interior point a_i of A_i is carried into point a. The vector $\overrightarrow{aa_i}$ is an integral vector. Consequently, there exist $n + 1$ interior points a_i of Q such that the vectors $\overrightarrow{aa_i}$ are all integral.

Let b be a vertex of the square R_1 which is closest to the point a (b is a node of the lattice). The length of the vector $\overrightarrow{ab}$ is less than 1. If the entire plane is translated by the vector $\overrightarrow{ab}$, then each of the $n + 1$ points a_i is carried into a point b_i where $\overrightarrow{bb_i} = \overrightarrow{aa_i}$. Then the $\overrightarrow{bb_i}$ are all integral vectors, and since b is a node of the lattice, all the points b_i are nodes. Thus after being translated by a vector of length less than 1, the figure Q covers $n + 1$ nodes of the lattice, and the theorem is proved.

THEOREM 3. *If the area of figure Q is not greater than* 1, *then it is possible to translate Q so that it will cover no nodes of the integral lattice.*

Proof. Let R denote a square whose sides are parallel to the axes and have length equal to some integer n. Square R can cover no more than n^2 nodes of the lattice. In fact, on a line parallel to the x-axis there can lie only n nodes of the lattice which are interior points of R. On the other hand, not more than n such lines can pass through interior points of R.

Let us consider first the case in which the area of Q is less than 1. Let R be a square containing Q and let Q_1 denote that part of R that is complementary to Q. Then the area of Q_1 is greater than $n^2 - 1$ (the area of R is n^2). By Blichfeldt's theorem it is possible to translate the plane by a vector of length less than 1 such that figure Q_1 goes into a figure $\overline{Q}_1$ covering $(n^2 - 1) + 1 = n^2$ nodes of the lattice. This translation takes R and Q into $\overline{R}$ and $\overline{Q}$. Since $\overline{R}$ can cover no more than n^2 nodes and part $\overline{Q}_1$ of it covers all n^2 of them, then the complementary part $\overline{Q}$ to $\overline{Q}_1$ of $\overline{R}$ covers no nodes.

The case in which the area of Q is equal to 1 is obtained by passing to limits, and the proof is completed.

43. TOPOLOGICAL THEOREMS OF LEBESGUE AND BOL'-BROUWER

Now we shall prove two topological theorems concerning convex figures and their homeomorphisms. One of these is the theorem proved in 1921 by the French mathematician Lebesgue (1875–1941) concerning properties of coverings of convex figures by closed sets. The other theorem is about fixed points under the transformations of a convex set into itself. It was first found in 1905 by the Latvian mathematician P. G. Bol' (1865–1921) as an auxiliary proposition to his analytical investigations. In 1911 it was rediscovered and formulated by the Dutch mathematician Brouwer. In 1924 the German mathematician Sperner gave a new proof of the theorem of Lebesgue based on two lemmas which are interesting in their own right. The Polish mathematicians Knaster, S. Mazurkiewicz, and Janiszewski have also reduced the proof of the Bol'-Brouwer theorem to the lemmas of Sperner.

We shall give proofs of the theorems of Lebesgue and Bol'-Brouwer based on the lemmas of Sperner for the two-dimensional case and indicate how to generalize the proof to the n-dimensional case.

Let us adopt the following definitions:

DEFINITION. *The* union *M of the sets $M_1, M_2, \ldots, M_n$ is the set of all points which lie in at least one of the M_i. We write*

$$M = M_1 \cup M_2 \cup \cdots \cup M_n.$$

We say that a set B is covered *by the sets $M_1, M_2, \ldots, M_n$ if B is contained in their union.*

Let us consider a transformation F of a set B into itself. To each point b of B there corresponds its *image* point $b_1 = F(b)$ of the same set. Point b is said to be *fixed* under the transformation F if it coincides with its image,

$$b = F(b).$$

DEFINITION. *A partition of a plane figure Q into triangles (rectilinear or curvilinear) is said to be a* simplicial partition *if two triangles of the partition can have in common only one vertex or one entire side.*

The vertices and sides of the triangles in the partition will be called the vertices and sides of the partition. We shall denote vertices and sides of a partition by integers; the equation $P = (k)$ means that vertex P of the partition is denoted by the integer k.

If P_1P_2 is a side and $P_1P_2P_3$ is a triangle of the partition and $P_1 = (k)$, $P_2 = (l)$, $P_3 = (m)$, then we shall write:

$$P_1P_2 = (k, l), P_1P_2P_3 = (k, l, m).$$

First lemma of Sperner. *Suppose that triangle $S = A_1A_2A_3$ is partitioned simplicially into triangles $\triangle_1, \triangle_2, \ldots, \triangle_k$ and each vertex of the partition is denoted by one of the numbers* 1, 2, 3, *such that*

(I) $A_1 = (1)$, $A_2 = (2)$, $A_3 = (3)$;

(II) *if a vertex P of the partition lies on a side A_iA_j of triangle S $(i, j = 1, 2, 3)$, then $P = (i)$* or *$P = (j)$.*

Then among the triangles of the partition there is some triangle $\triangle_i = (1, 2, 3)$.

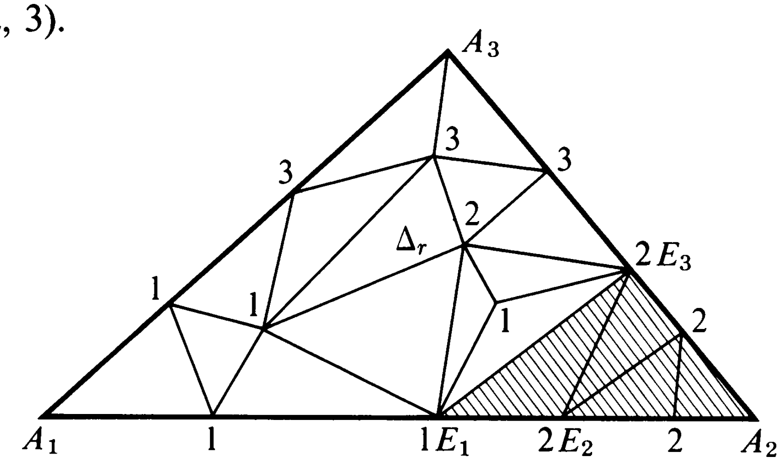


Fig. 174

Proof. The proof is by induction on the number, k, of triangles in the partition. If $k = 1$, then the only triangle in the partition is $S = A_1A_2A_3 = (1, 2, 3)$. So the lemma is true for $k = 1$.

Now we assume the proposition is true for all $k < m$, and we shall show that it is true for $k = m$. Then it will be proved for arbitrary k. Suppose triangle $S = A_1A_2A_3$ is partitioned simplicially into triangles $\triangle_1, \triangle_2, \ldots, \triangle_m$. On the side A_1A_2 of triangle S lies a finite sequence of vertices (1) and (2) of the partition starting with $A_1 = (1)$ and $A_2 = (2)$. Among these there is an adjacent pair of vertices E_1 and E_2 such that $E_1 = (1)$ and $E_2 = (2)$ (Fig. 174). E_1E_2 is a side of the partition belonging to some triangle $\triangle_i = E_1E_2E_3$ of the partition. If $E_3 = (3)$, then $\triangle_i = (1, 2, 3)$ and we have found the desired triangle.

Let us consider the case $E_3 = (2)$. (The case $E_3 = (1)$ is completely analogous.) E_3 lies either on side A_2A_3 or inside S.

First, let us suppose that $E_3 = (2)$ lies on A_2A_3 (Fig. 174). E_3 coincides neither with $A_3 = (3)$ nor with A_2. Let us remove from S the triangle $E_1A_2E_3$ (shaded in Figure 174). We denote the remaining part of S by Q and consider it as a curvilinear triangle with vertices $A_1' = A_1 = (1)$, $A_2' = E_3 = (2)$, $A_3' = A_3 = (3)$ and sides $A_1'A_3' = A_1A_3$, $A_2'A_3' = E_3A_3$ (part of A_2A_3), and $A_1'A_2'$, the broken line $A_1E_1E_3$. The triangles $\triangle_j$ of the partition that lie inside Q form a simplicial partition of Q, and the number k of them is smaller than m. Retaining the original denotations 1, 2, and 3 for the vertices of the partition, we see that conditions (I) and (II) of the lemma are satisfied by the new triangle Q. $A_i' = (i)\,(i = 1, 2, 3)$; that is, condition (I) is satisfied, and the vertices found on side $A_i'A_j'$ belong to side A_iA_j of triangle S, with the exception of vertex $A_2' = E_3 = (2)$ on side $A_1'A_2'$, and condition (II) is also satisfied.

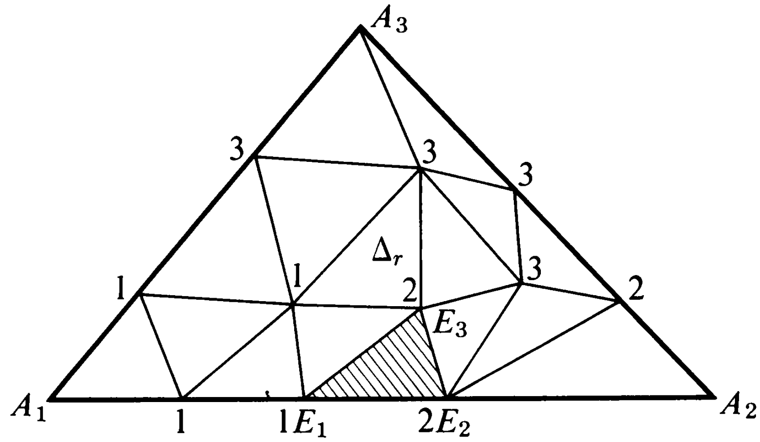


Fig. 175

Next, let us suppose that $E_3 = (2)$ lies inside S (Fig. 175). We let Q denote the figure obtained from S by removing the triangle $\triangle_j = E_1E_2E_3$ from S. We shall consider Q as a triangle having for its vertices A_i' $(i = 1, 2, 3)$ the previous vertices $A_i = (i)$, and for sides $A_2'A_3'$ and $A_1'A_3'$ the previous sides A_2A_3 and A_1A_3, and for side $A_1'A_2'$ the broken line $A_1E_1E_3E_2A_2$. The triangles $\triangle_i$ of the partition that lie inside Q form a simplicial partition of Q, and their number is $m - 1 < m$. Retaining the original denotations for the vertices of the partition, we see that Q satisfies conditions (I) and (II) of the lemma: $A_i' = A_i = (i)$, and all vertices found on $A_i'A_j'$ lie on $A_iA_j\,(i, j = 1, 2, 3)$ with the exception of vertex $E_3 = (2)$ lying on $A_1'A_2'$.

Thus in both cases the number of triangles $\triangle_i$ in the partition of triangle Q is less than m and the conditions of the lemma are satisfied. Then by the induction hypothesis we can find among the triangles $\triangle_i$ covering Q some triangle $\triangle_r = (1, 2, 3)$.

Therefore such a triangle must exist in all cases and the lemma is proved.

SECOND LEMMA OF SPERNER. *Suppose that triangle $Q = A_1A_2A_3$ is covered by three closed sets R_1, R_2, R_3* (Fig. 176) *such that*

(I) *A_1 belongs to R_1, A_2 to R_2, and A_3 to R_3;*

(II) *each side A_iA_j $(i, j = 1, 2, 3)$ is covered by the set $R_i \cup R_j$.*
Then there exists some point in the triangle which belongs to all three sets R_1, R_2, and R_3; that is, their intersection is not empty.

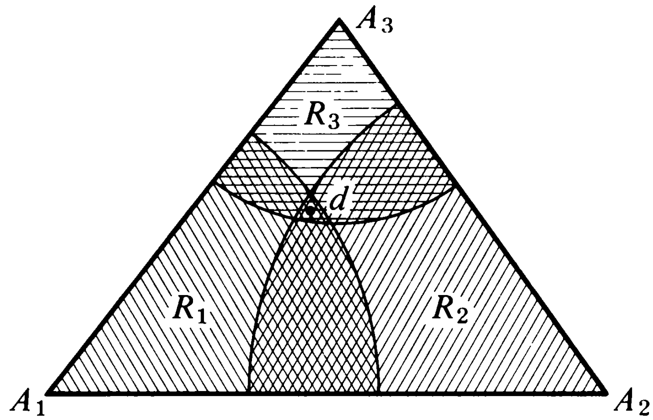


Fig. 176

Proof. Select a sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots$ of positive numbers approaching zero as a limit, and for each ϵ_n find a simplicial partition of Q into triangles $\triangle_i^{(n)}$ of diameter less than ϵ_n. For each n we shall denote the vertices of the nth partition by the numbers 1, 2, 3 such that if $P = (i)$, then P belongs to the corresponding set R_i $(i = 1, 2, 3)$. (If P belongs to two or three such sets, then P may be denoted by any one of the two or three indices of the sets containing it.) In doing this, we shall be sure to denote A_i $(i = 1, 2, 3)$ by the index i. (In any case, A_i belongs to R_i.) A vertex lying on side A_iA_j will be denoted by one of the two indices i or j. (This vertex must belong to at least one of the two sets R_i and R_j.) This denotation satisfies conditions (I) and (II) of the first lemma of Sperner. Hence there exist triangles $\triangle_r^{(n)} = (1, 2, 3)$ with vertices $a_n = (1)$, $b_n = (2)$, $c_n = (3)$. These three points are separated from each other by distances smaller than ϵ_n, and a_n belongs to R_1, b_n to R_2, and c_n to R_3.

We have three sequences of points: $\{a_n\}$ of R_1, $\{b_n\}$ of R_2, and $\{c_n\}$ of R_3 ($n = 1, 2, 3, \ldots$). Since all the points $\{a_n\}$ lie in triangle Q, then this sequence has a limit point d in Q. As n increases, the points b_n and c_n become unboundedly close to a_n, so that the limit point d of the sequence $\{a_n\}$ is also a limit point of the sequences $\{b_n\}$ and $\{c_n\}$. The sets R_1, R_2, and R_3 are closed, and point d is a limit point of the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ of points of R_1, R_2, and R_3 respectively. Therefore point d belongs to all three of these sets, and the lemma is proved.

LEBESGUE'S THEOREM. *Given an equilateral triangle $S = A_1A_2A_3$ and an arbitrary positive number ϵ smaller than half the altitude of the triangle S, if triangle S is covered by a system of closed sets Q_1, Q_2, $\ldots$, Q_n each having a diameter smaller than ϵ, then in S there exists a point d belonging to at least three of these sets* (Fig. 177).

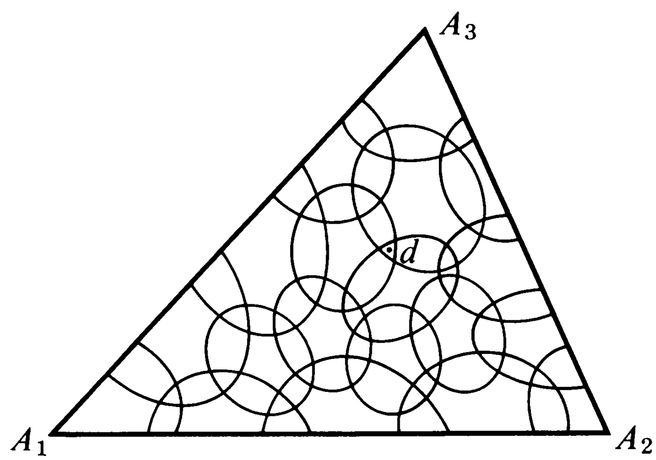


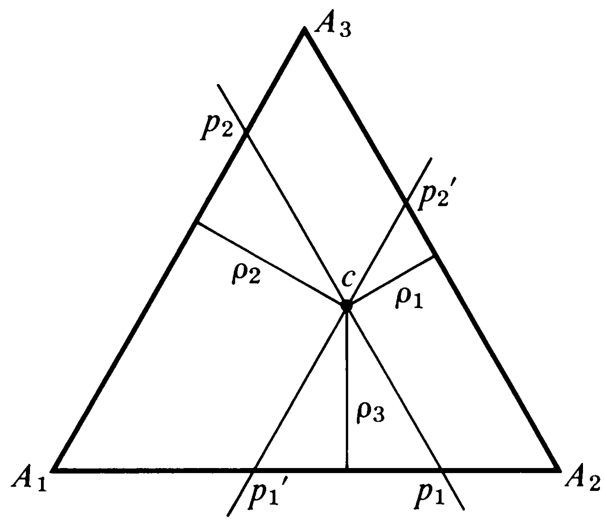
Fig. 177

Proof. We shall divide the sets $Q_1, Q_2, \ldots, Q_n$ into three groups. The set Q_i containing point A_1 belongs to group I, the set containing A_2 belongs to group II, and the set containing A_3 belongs to group III (no set Q_i contains two of the points A_i). Of the remaining sets, those containing points of side A_1A_2 belong to group I or II, those containing points of side A_2A_3 belong to group II or III, and those containing points of side A_1A_3 belong to group I or III. Since the diameter of each set Q_i is smaller than half the altitude of S, Q_i cannot have points in common with all three sides of S. If Q_i has common points with the two sides A_1A_2 and A_1A_3, then Q_i belongs to group I; if with A_2A_3 and A_1A_2, then to group II; if with A_1A_3 and A_2A_3, then to group III. The remaining sets may belong to any of the three groups.

Let R_1, R_2, R_3 denote the unions of the sets in groups I, II, and III respectively. The set R_i contains vertex A_i ($i = 1, 2, 3$) and contains no points of the opposite side; the side A_iA_j is covered by the union of the sets R_i and R_j ($i, j = 1, 2, 3$); finally, the entire triangle is the union of the sets R_1, R_2, R_3. By the Second Lemma of Sperner there exists a common point d of these sets. Since d belongs to R_1, it belongs to some set Q_i of group I. Analogously, because d belongs to R_2 and R_3, it belongs to some sets Q_i of groups II and III. Therefore d belongs to three of the sets Q_i, and the theorem is proved.

THE BOL'-BROUWER THEOREM. *If F is a continuous transformation of an equilateral triangle $S = A_1A_2A_3$ into itself, then there exists a point b of S that is carried into itself by F (such a point is called a fixed point of the transformation): $b = F(b)$.*

Fig. 178



Proof. Let c be an arbitrary point of S. Let ρ_1, ρ_2, ρ_3 be its distances from the sides A_2A_3, A_1A_3, A_1A_2 of the triangle. We have

$$\rho_1 + \rho_2 + \rho_3 = \rho, \tag{1}$$

where ρ is a constant. For if the length of a side of S is a, then the area of S is $\frac{1}{4}a^2\sqrt{3}$. Let us divide S (Fig. 178) into three triangles A_2A_3c, A_1A_3c, and A_1A_2c. Their areas are $\frac{1}{2}a\rho_1$, $\frac{1}{2}a\rho_2$, and $\frac{1}{2}a\rho_3$ from which it follows that

$$\frac{a}{2}(\rho_1 + \rho_2 + \rho_3) = \frac{a^2\sqrt{3}}{4}.$$

Therefore

$$\rho_1 + \rho_2 + \rho_3 = \frac{a\sqrt{3}}{2} = \rho.$$

The line

$$\rho_1 = \text{constant}$$

is a line p_1p_2 parallel to side A_2A_3, lying on the same side of it as A_1, and separated from A_2A_3 by the distance ρ_1. Analogously the line

$$\rho_2 = \text{constant}$$

is a line $p_1'p_2'$ parallel to A_1A_3, lying on the same side of it as A_2, and separated from A_1A_3 by the distance ρ_2 (Fig. 178). Therefore, if the values $\rho_1 = h_1$ and $\rho_2 = h_2$ are known for point c (and, hence, so is $\rho_3 = \rho - \rho_1 - \rho_2$), then they uniquely determine the point c, for c lies at the intersection of the lines $\rho_1 = h_1$ and $\rho_2 = h_2$.

If point c is moved, then the values of ρ_1, ρ_2, and ρ_3 are changed, but their sum remains constant by virtue of (1). Hence, ρ_1, ρ_2, and ρ_3 cannot all three simultaneously increase; at least one of them must not increase. If point c lies on side A_1A_2, then $\rho_3 = 0$ and $\rho_1 + \rho_2 = \rho$. In general, if c lies on side A_iA_j, then $\rho_i + \rho_j = \rho$ $(i, j = 1, 2, 3)$. If c is moved, the sum $\rho_i + \rho_j$ cannot increase. Hence at least one of ρ_i and ρ_j does not increase. For $c = A_1$, $\rho_2 = \rho_3 = 0$, $\rho_1 = \rho$, and ρ_1 cannot increase if c is moved. In general, ρ_i cannot increase as $c = A_i$ is moved away from A_i $(i = 1, 2, 3)$.

We shall let R_i $(i = 1, 2, 3)$ denote the set of all points of triangle S for which the value of ρ_i does not increase under the continuous transformation F. Each point of the triangle belongs to one of these sets. Each vertex A_i belongs to the set R_i. Each point of side A_iA_j belongs to one of the sets R_i and R_j.

The sets R_1, R_2, R_3 are closed. This is true because our transformation F is continuous. For example, let point b be a limit point of set R_1, that is, of points for which the value of ρ_1 does not increase under the transformation (points not moving away from A_2A_3). Then the value of ρ_1 for point b also does not increase (b does not move away from A_2A_3). Hence, b belongs to R_1, and R_1 contains all of its limit points; that is, R_1 is closed.

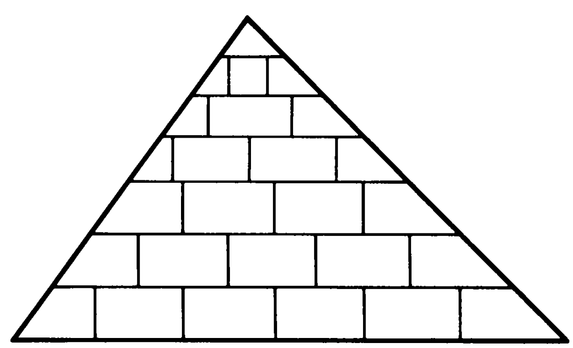
By the Second Lemma of Sperner the sets R_1, R_2, and R_3 have some point b in common. For this point the values of ρ_1, ρ_2, and ρ_3 do not increase under the transformation F. If one of these values were to decrease, then their sum $\rho_1 + \rho_2 + \rho_3$ would also decrease. But this sum is constant. Thus the values of ρ_1, ρ_2, and ρ_3 for point b do not change under our transformation. Hence, point b does not move under this transformation, and the theorem is proved.

The theorems of Lebesgue and of Bol'-Brouwer remain valid if we replace triangle S by any plane convex figure R. This is proved by forming a topological transformation of S onto R. Sufficiently small closed sets covering S are carried into sufficiently small closed sets covering R, and a common point of three of these closed sets on S is carried into a common point of the corresponding three closed sets on R. We obtain the following formulation of Lebesgue's theorem:

If a convex plane figure (for example, a square) is covered by a system of sufficiently small (having small diameters) closed sets Q_1, $Q_2, \ldots, Q_r$, then there exists at least one point that is common to three of these sets.

We may supplement Lebesgue's theorem by noting that a plane convex figure R (for example, a triangle) may be covered by a system of arbitrarily small closed sets $Q_1, Q_2, \ldots, Q_n$ such that no point of R belongs to more than three of these sets. The proof is contained in Figure 179.

Fig. 179



The Bol'-Brouwer theorem in more general formulation reads:

A continuous transformation of any convex figure R into itself has at least one fixed point.

44. GENERALIZATION TO *n* DIMENSIONS

The role which the triangle plays in the plane and which the tetrahedron plays in three-dimensional space is played in n-dimensional space by the *n-dimensional simplex:*

DEFINITION. *Let $A_0, A_1, A_2, \ldots, A_n$ be points in an n-dimensional space not lying in an $(n-1)$-dimensional subspace. The smallest convex solid containing these points (the convex hull of the points A_0, A_1, $A_2, \ldots, A_n$) is called the* n-dimensional simplex *determined by them.*

The n-dimensional simplex is the simplest of the n-dimensional polyhedra. For $k = 0, 1, 2, \ldots, n - 1$ the n-dimensional simplex has k-dimensional faces, which are the k-dimensional simplexes determined by all possible groups of $k + 1$ vertices among the $n + 1$ vertices $A_0, A_1, A_2, \ldots, A_n$. The number of these is $\binom{n+1}{k+1}$. Thus, an n-dimensional simplex has $n + 1 = \binom{n+1}{1}$ vertices $(k = 0)$, $\frac{(n+1)n}{1\cdot 2} = \binom{n+1}{2}$ edges $(k = 1)$, $\frac{(n+1)n(n-1)}{1\cdot 2\cdot 3} = \binom{n+1}{3}$ plane faces $(k = 2), \ldots, n + 1 = \binom{n+1}{n}$ $(n - 1)$-dimensional faces. For $n = 1, 2$, and 3 the simplexes are the line segment, the triangle, and the tetrahedron.

A *partition* of an n-dimensional polyhedron into n-dimensional simplexes is said to be *simplicial* if the common part of any two simplexes in the partition is *one entire k-dimensional face* (for some $k = 0, 1, 2, \ldots, n - 1$), that is, one vertex, or one entire edge, . . . , or one entire $(n - 1)$-dimensional face.

First lemma of Sperner. *Let an n-dimensional simplex with vertices $A_1, A_2, \ldots, A_{n+1}$ be partitioned simplicially into simplexes $\triangle_1, \triangle_2, \ldots, \triangle_k$ and the vertices of the partition denoted by the numbers $1, 2, \ldots, n + 1$ such that the following conditions are satisfied:*

(I) *each vertex A_i is denoted by the integer i: $A_i = (i)$ $(i = 1, 2, \ldots, n + 1)$;*

(II) *a vertex of the partition lying in a k-dimensional face determined by the vertices $A_{i_0}, A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ is denoted by one of the numbers $i_0, i_1, i_2, \ldots, i_k$.*

Then there exists a simplex $\triangle_i$ in the partition whose vertices are denoted by all the numbers $1, 2, \ldots, n + 1$.

From this lemma we can prove the following in the same way as in the two-dimensional case.

Second lemma of Sperner. *Suppose that an n-dimensional simplex with vertices $A_1, A_2, \ldots, A_{n+1}$ is covered by closed sets $R_1, R_2, \ldots, R_{n+1}$ so that the following conditions are satisfied:*

(I) *the vertex A_i $(i = 1, 2, \ldots, n + 1)$ belongs to the set R_i;*

(II) *the k-dimensional face $(k = 1, 2, \ldots, n - 1)$ with vertices $A_{i_0}, A_{i_1}, \ldots, A_{i_k}$ belongs to the union $R_{i_0} \cup R_{i_1} \cup \cdots \cup R_{i_k}$.*

Then there exists a point of the simplex belonging to all $n + 1$ sets $R_1, R_2, \ldots, R_{n+1}$ (i.e., the intersection of these sets is nonempty).

The theorems of Lebesgue and Bol'-Brouwer have also been proved for the n-dimensional case.

LEBESGUE'S THEOREM. *If an n-dimensional cube is covered by a finite system of sufficiently small closed sets, then at least one point of the cube is contained in $n + 1$ of these sets.*

Also, an n-dimensional cube may be covered by a finite system of arbitrarily small closed sets so that no point belongs to more than $n + 1$ of the sets.

THE BOL'-BROUWER THEOREM. *Any continuous transformation of a convex n-dimensional solid into itself has a fixed point.*

These theorems for the n-dimensional case are proved from the lemmas of Sperner in the same way as for the two-dimensional case.

EXAMPLE. Let R be the square in the plane consisting of all points (x, y) satisfying the inequalities $|x| \leq 1$, $|y| \leq 1$. To each point $b(x, y)$ of R there corresponds the point $b_1(x_1, y_1)$, $b_1 = F(b)$, where

$$x_1 = \frac{1}{2}\sin(x + y) + \frac{1}{4}2^x,$$

$$y_1 = \frac{1}{2}(x^3 + \cos y).$$

Clearly, $|x_1| \leq 1$, $|y_1| \leq 1$; that is, the point $b_1(x_1, y_1) = F(b)$ belongs to R. Thus F is a continuous transformation of the square R into itself. By virtue of the Bol'-Brouwer theorem there exists a fixed point $b(x, y)$ of this transformation:

$$b(x, y) = b_1(x_1, y_1);$$

that is, $x = x_1$, $y = y_1$. The coordinates x, y of point b satisfy the equations

$$\left.\begin{aligned} x &= \frac{1}{2}\sin(x + y) + \frac{1}{4}2^x, \\ y &= \frac{1}{2}(x^3 + \cos y). \end{aligned}\right\} \tag{1}$$

This system of equations has a solution x, y for which the point $b(x, y)$ belongs to R, that is, $|x| \leq 1$, $|y| \leq 1$. It would be difficult to show that system (1) has such a solution by using ordinary methods.

This example illustrates the use of the Bol'-Brouwer theorem in establishing the existence of solutions of systems of equations.

45. CONVEX FIGURES IN NORMED SPACES

Let us consider the vectors drawn from some point O in a plane or space. The letters a, b, . . . will be used both to denote points in the plane (or space) and the vectors $\overrightarrow{Oa}, \overrightarrow{Ob}, \ldots$ terminating in these points. Vectors are added according to the parallelogram rule. Hence the length of a vector $a + b$ cannot exceed the sum of the lengths of the vectors a and b (Fig. 180).

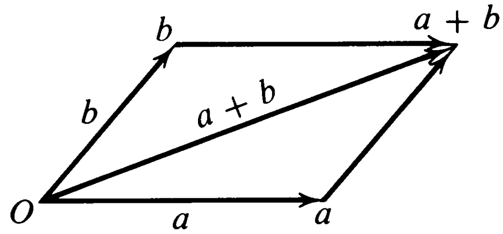
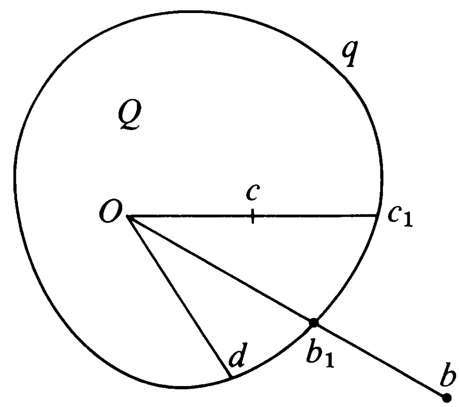


Fig. 180

Let us consider a plane convex figure Q with the origin O as an interior point and boundary q. Let b and c be arbitrary points of the plane ($\overrightarrow{Ob}$ and $\overrightarrow{Oc}$ are arbitrary vectors). We shall let b_1 and c_1 denote the points of intersection of the half-lines Ob and Oc with q (Fig. 181).

Fig. 181



DEFINITION. *The ratio of the segments Ob and Ob_1 is called the* norm *of the vector b and is denoted by the symbol* $\|b\|$.

For points b lying outside Q, $\|b\| > 1$. For points c lying inside Q, $\|c\| < 1$, and if $c \neq O$, then $\|c\| > 0$. For points d lying on the boundary q, $\|d\| = 1$. For the origin O we write $\|O\| = 0$.

Norms satisfy the following conditions:

(i) The norm of any point (vector) a different from O is positive:

$$\|a\| > 0, \quad \text{and only} \quad \|O\| = 0.$$

(ii) If s is a positive number, then

$$\|sa\| = s\|a\|.$$

(iii) We shall show that norms satisfy the triangle inequality:

$$\|a + b\| \leq \|a\| + \|b\|.$$

First we shall prove:

If a_0 *and* a_1 *are two points lying on the boundary* q, *and* t_0, t_1 *are arbitrary positive numbers, then*

$$\|t_0 a_0 + t_1 a_1\| \leq t_0 + t_1. \tag{1}$$

Proof. Notice that

$$\begin{aligned} t_0 a_0 + t_1 a_1 &= (t_0 + t_1)\left(\frac{t_0}{t_0 + t_1} a_0 + \frac{t_1}{t_0 + t_1} a_1\right) \\ &= (t_0 + t_1)(s_0 a_0 + s_1 a_1), \end{aligned} \tag{2}$$

where

$$s_0 = \frac{t_0}{t_0 + t_1} > 0, \quad s_1 = \frac{t_1}{t_0 + t_1} > 0, \quad s_0 + s_1 = 1.$$

The point $(s_0 a_0 + s_1 a_1)$ lies on the segment $a_0 a_1$, and so together with a_0 and a_1 it belongs to convex figure Q. Therefore

$$\|s_0 a_0 + s_1 a_1\| \leq 1,$$

and by means of equation (2) we prove (1):

$$\|t_0 a_0 + t_1 a_1\| = (t_0 + t_1)\ \|s_0 a_0 + s_1 a_1\| \leq (t_0 + t_1).$$

Proof of (iii). Now let a and b be arbitrary points (vectors) different from O, and let a_0 and b_0 be the intersections of half-lines Oa and Ob with the boundary q of figure Q. By virtue of the definition of the norm, if

$$\|a\| = t_0, \qquad \|b\| = t_1,$$

then

$$a = t_0 a_0, \qquad b = t_1 b_0.$$

From (1) and the above, we obtain

$$\|a + b\| = \|t_0 a_0 + t_1 b_0\| \leq (t_0 + t_1) = \|a\| + \|b\|,$$

and (iii) is proved.

If one of the vectors a and b, for example a, is the zero vector $a = O$, then $a + b = b$, $\|b\| = \|a + b\|$, $\|a\| = 0$. Hence, inequality (iii) is also valid in this case.

Whereas we have just defined the norm in terms of some convex figure Q, we shall now do just the opposite. Given a norm satisfying certain conditions, we shall construct Q.

Suppose that to each vector a, there corresponds a number $\|a\|$ satisfying the properties of a norm:

(i) $\|a\| \geq 0$ *and if* $a \neq O$, *then* $\|a\| > 0$;

(ii) *for* $s \geq 0$, $\|sa\| = s\|a\|$;

(iii) $\|a + b\| \leq \|a\| + \|b\|$.

Then the set Q of all vectors a satisfying the condition $\|a\| \leq 1$ *is a convex figure.*

Proof. Let a and b belong to Q, that is, $\|a\| \leq 1$, $\|b\| \leq 1$. We shall show that Q contains the entire line segment connecting a and b, that is, all points of the form

$$sa + s_1 b \quad (s > 0, s_1 > 0; s + s_1 = 1).$$

By condition (iii)

$$\|sa + s_1 b\| \leq \|sa\| + \|s_1 b\|.$$

Further, by condition (ii)

$$\|sa\| = s\|a\| \leq s, \qquad \|s_1 b\| = s_1\|b\| \leq s_1.$$

Hence,

$$\|sa + s_1 b\| \leq s + s_1 = 1;$$

that is, $(sa + s_1 b)$ also belongs to Q, and Q is convex.

Q is called the *unit sphere.* Hence the convexity of the unit sphere is equivalent to the triangle inequality.

We have the same situation in three-dimensional space, and we may also study norms for points (vectors) in an n-dimensional space.

Definition. *A space with a norm satisfying conditions* (i)–(iii) *is called a* normed space.

Now we shall give a few examples of two-dimensional normed spaces.

Example 1. For $a(x, y)$, let

$$\|a\| = \sqrt{x^2 + y^2};$$

that is, the norm of a vector coincides with its ordinary length. The unit sphere Q is an ordinary circle with radius 1.

EXAMPLE 2. For $a = (x, y)$, let $\|a\| = |x| + |y|$. Then the unit sphere Q is the set of points (x, y) for which $|x| + |y| \leq 1$. On the boundary of this unit sphere $|x| + |y| = 1$ (Fig. 182).

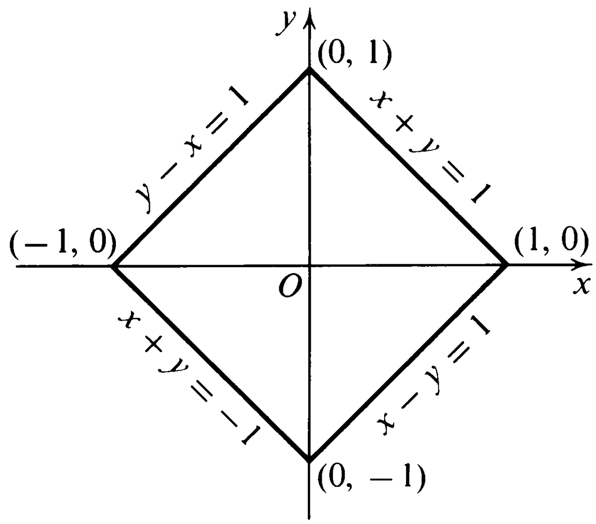


Fig. 182

The boundary of Q in the first quadrant (where x and y are positive) is a part of the line $x + y = 1$. In the second quadrant, where $x < 0, y > 0$, it is a part of the line $y - x = 1$. In the third, where x and y are negative, part of the line $-y - x = 1$ or $x + y = -1$. Finally, in the fourth quadrant, where x is positive and y negative, part of the line $x - y = 1$. Thus Q is a square with vertices at the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$.

G. Minkowski has studied normed n-dimensional spaces in connection with the theory of convex solids.

So-called Banach spaces, named after the Polish mathematician S. Banach (1892–1945), play a large role in contemporary mathematics. Elements of such spaces may be added to each other and multiplied by constant numbers as are vectors, and we can introduce the concept of a norm for them satisfying conditions (i)–(iii). We shall give a few examples of *Banach spaces*.

EXAMPLE 3. Consider the set C of all continuous functions y defined on the segment $0 \leq x \leq 1$. Addition of such functions and multiplication of them by real numbers is defined in the usual way. The norm $\|y\|$ of the function y is defined as the maximum absolute value $|y(x)|$ of y on the segment $0 \leq x \leq 1$. For example, if $y = x^2$, then $\|y\| = 1$; if $y = 2 - x$, then $\|y\| = 2$.

The role of the zero element is played by the function $y_0 = 0$, which equals zero everywhere, and it is easy to show that the norm $\|y\|$ satisfies conditions (i)–(iii) and that C is a Banach space.

EXAMPLE 4. Consider the set of all continuous functions y on the segment $0 \leq x \leq 1$. Let us take as the norm of the function y the value

$$\|y\| = \sqrt{\int_0^1 y^2\,dx}.$$

It is obvious that conditions (i) and (ii) are satisfied. We shall prove that this norm also satisfies (iii):

$$\sqrt{\int_0^1 (y + z)^2\,dx} \leq \sqrt{\int_0^1 y^2\,dx} + \sqrt{\int_0^1 z^2\,dx} \tag{3}$$

or the equivalent inequality

$$\int_0^1 (y + z)^2\,dx \leq \left[\sqrt{\int_0^1 y^2\,dx} + \sqrt{\int_0^1 z^2\,dx}\right]^2. \tag{4}$$

We have

$$\begin{aligned}\int_0^1 (y + z)^2\,dx &= \int_0^1 (y^2 + 2yz + z^2)\,dx \\ &= \int_0^1 y^2\,dx + 2\int_0^1 yz\,dx + \int_0^1 z^2\,dx,\end{aligned} \tag{5}$$

$$\begin{aligned}&\left[\sqrt{\int_0^1 y^2\,dx} + \sqrt{\int_0^1 z^2\,dx}\right]^2 \\ &\qquad = \int_0^1 y^2\,dx + 2\sqrt{\int_0^1 y^2\,dx \int_0^1 z^2\,dx} + \int_0^1 z^2\,dx.\end{aligned} \tag{6}$$

Let us compare the right-hand parts of equations (5) and (6). Recall the Schwarz inequality

$$\int_a^b yz\,dx \leq \sqrt{\int_a^b y^2\,dx \int_a^b z^2\,dx}.$$

From it and equations (5) and (6) follows inequality (4) and, hence, (3). Thus the triangle inequality is valid for the norm we have introduced.

The norm in Example 4 resembles the norm for ordinary Euclidean space. The class of functions (not only continuous, but also a broad class of discontinuous functions) with this norm forms a function space called a *Hilbert space.* It plays a fundamental role in mathematical analysis.

With the introduction of function spaces (spaces whose elements are functions) into mathematics there arose a new branch of analysis, *functional analysis,* broadly applying common geometric considerations in function spaces.

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